Studies in Absolute Irreducibility over Finite Fields: Rational Points, Singular Zeta-Functions, APN Functions, and Exceptional Polynomials

THESIS
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Abstract

A central problem in algebraic geometry is to determine when a given curve is absolutely irreducible over its ground field. The absolute irreducibility of a given curve also has many applications in: algebraic geometry, coding theory, cryptography, number theory, combinatorics, and the study of finite fields. Absolute irreducibility can be used for point counting on algebraic curves. Counting the rational points on a given curve leads to the study of zeta functions. On the side of coding theory, there exists an intrinsic relationship between the cyclic codes $C_s^{(t)}$ and the curve defined by the polynomial:

$$\phi_t(x,y,z) = \frac{x^t + y^t + z^t + (x+y+z)^t}{(x+y)(y+z)(x+z)}$$

The error-correction capability of $C_s^{(t)}$ depends on the number of rational points with distinct coordinates on $\phi_t(x,y,z)$. We extend an existing table of rational points on $\phi_t(x,y,z)$ to the range $2 \leq s \leq 20$ and $3 \leq t \leq 255$ (with t odd). We study the rational points of $\phi_{15}(x,y,z)$ and its zeta function. We can also use these codes to construct LDPC codes.

We generalize $\phi_t(x,y,z)$ to the polynomial:

$$\phi_f(x,y,z) = \frac{f(x) + f(y) + f(z) + f(x+y+z)}{(x+y)(y+z)(x+z)}$$

and use it to study almost perfect non-linear (APN) trinomial functions having the form $f(x) = x^{2^i e} + h(x)$, and whether or not they are exceptional APN. The computation of these functions aids in the resolution of the exceptional APN conjecture.

Finally, we shift our focus to the study of permutation and exceptional polynomials defined over $\mathbb{F}_q$. We study them via the curve defined by:

$$\Gamma_f(x,y) = \frac{f(x) - f(y)}{x - y}$$

we use absolute irreducibility criterion on $\Gamma_f(x,y)$ over the quadratic extension $\mathbb{F}_{q^2}$ to determine which $f(x)$ are exceptional polynomials.