Foundamental Solutions to Time fractional Poisson equations

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Generator

Given a Markov process $(X, \mathbb{P}_x, x \in E)$ on E, its transition semigroup $\{P_t; t \geq 0\}$ is given by

$$P_t\phi(x) = \mathbb{E}_x[\phi(X_t)].$$

The infinitesimal generator $\mathcal{L}$ of X is

$$\mathcal{L}\phi(x) = \lim_{t \to 0} \frac{P_t\phi(x) - \phi(x)}{t}.$$

Hence $u(t, x) = P_t\phi(x)$ solves $\frac{\partial u}{\partial t} = \mathcal{L}u$ with $u(0, x) = \phi(x)$.

- When X is Brownian motion on $\mathbb{R}^d$, $\mathcal{L} = \frac{1}{2}\Delta$.
- When X is an absorbing (or reflecting) Brownian motion in $D \subset \mathbb{R}^d$, $\mathcal{L}$ is the Dirichlet (or Neumann Laplacian in D.
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Dirichlet boundary value problem

Note that $P_t\phi(x) - \phi(x) = \int_0^t \mathcal{L} P_s \phi(x) dt$. If $\phi \in \mathrm{Dom}(\mathcal{L})$, then

$$P_t\phi(x) - \phi(x) = \int_0^t P_s \mathcal{L}\phi(x) dt.$$

In other words, $\mathbb{E}_x M_t = 0$ for $t \geq 0$ and $x \in E$, where

$$M_t = \phi(X_t) - \phi(X_0) - \int_0^t \mathcal{L}\phi(X_s) ds.$$

This implies that M_t is a martingale.

Define $\tau_D = \inf\{t \geq 0 : X_t \notin D\}$. Then (under suitable condition) $\mathbb{E}_x M_{\tau_D} = 0$. So if $\mathcal{L}\phi = 0$ in D with $\phi(x) = g(x)$ on D^c, then

$$\phi(x) = \mathbb{E}_x \left[g(X_{\tau_D}) \right], \quad x \in D.$$

S. Kakutani (1944): used Brownian motion to solve classical Dirichlet boundary value problem.

Poisson equation

Under suitable conditions, the solution to

$$\frac{\partial u}{\partial t} = \mathcal{L}u + f(t,x) \quad \text{with } u(0,x) = \phi(x)$$

is given by

$$\begin{aligned} u(t,x) &= P_t\phi(x) + \int_0^t P_{t-s}f(s,\cdot)(x)ds \\ &= \mathbb{E}_x\phi(X_t) + \mathbb{E}_x \int_0^t f(t-s, X_s)ds. \end{aligned}$$

Why? Formally,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}P_t\phi(x) + P_0 f(t,\cdot)(x) + \int_0^t \frac{\partial}{\partial t}P_{t-s}f(s,\cdot)(x)ds \\ &= \mathcal{L}P_t\phi(x) + f(t,x) + \int_0^t \mathcal{L}P_{t-s}f(s,\cdot)(x)ds \\ &= \mathcal{L}u(t,\cdot)(x) + f(t,x). \end{aligned}$$

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Time fractional Poisson equation

The goal of this talk is to study

$$\partial_t^w v = \mathcal{L} v + f(t, x),$$

where

$$\partial_t^w g(t) := \frac{d}{dt} \int_0^t w(t-s)(g(s) - g(0)) ds.$$

Here $w \geq 0$ is a decreasing function with $w(0) = \infty$ and $\int_0^\infty (t \wedge 1)(-dw(t)) < \infty$.

When $w(r) = \frac{1}{\Gamma(1-\beta)} r^{-\beta}$ with $0 < \beta < 1$, $\partial_t^w g(t)$ is the classical Caputo fractional derivative ∂_t^β of order β.

Why do we care

• Non-local spatial operators can be used to model anomalous superdiffusion that describe particles move faster than Brownian motion (e.g. the random walker remains in motion without changing direction for a time that follows a Pareto-Lévy distribution).

• Fractional time equation has a close connection to anomalous subdiffusions that describe particles move slower than Brownian motion (or the original underlying spatial motion), for example, due to particle sticking and trapping.

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Subordinate Markov Process

Suppose $S = \{S_t; t \geq 0\}$ is a subordinator independent of X with Laplace exponent ϕ:

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}.$$

There is a unique $\kappa \geq 0$ and a measure $\nu(dx)$ with $\int_0^\infty (1 \wedge x)\nu(dx) < \infty$ so that

$$\phi(\lambda) = \kappa\lambda + \int_0^\infty (1 - e^{-\lambda x})\nu(dx).$$

X_{S_t} is a Markov process, called subordinate Markov process. When X is symmetric, the infinitesimal generator of X_{S_t} is $\mathcal{L}_\phi := -\phi(-\mathcal{L})$.

Hence $u(t,x) := \mathbb{E}_x[f(X_{S_t})]$ solves $\frac{\partial u}{\partial t} = \mathcal{L}_\phi u$ with $u(0,x) = f(x)$.

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Example: stable subordinator

When $\{S_t; t \geq 0\}$ is a β-subordinator with $0 < \beta < 1$ with Laplace exponent $\phi(\lambda) = \lambda^\beta$, the infinitesiaml generator of X_{S_t} is $-(-\mathcal{L})^\beta$.

When X is Brownian motion in $\mathbb{R}^d$, X_{S_t} is a rotationally symmetric (2β)-stable process in $\mathbb{R}^d$, whose infinitesimal generator is $-(-\Delta)^\beta =: \Delta^\beta$. It can also be expressed as

$$\begin{aligned}
\Delta^\beta f(x) &= \lim_{\varepsilon \to 0} \int_{\{|y-x|>\varepsilon\}} (f(y) - f(x)) \frac{c(d,\alpha)}{|y-x|^{d+2\beta}} dy \\
&= \int_{\mathbb{R}^d} (f(x+z) - f(x) - \nabla f(x) \cdot z 1_{\{|z|\leq 1\}}) \frac{c(d,\alpha)}{|z|^{d+2\beta}} dz.
\end{aligned}$$

Space dependent non-local operator: for foundamental solutions

- Symmetric case: C.-Kumagai 2003, 2008, 2010, $\cdots$
- Non-symmetric: C.-Zhang 2016, 2018, $\cdots$

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Suppose that $\mathcal{L}$ is the generator of a strong Markov process X.

Theorem (Baeumer-Meerschaert, 2001; Meerschaert-Scheffler, 2004): $u(t,x) = \mathbb{E}_x[f(X_{E_t})]$ solves

$$\frac{\partial^\beta u(t,x)}{\partial t^\beta} = \mathcal{L}_x u(t,x), \quad u(0,x) = f(x).$$

Here $E_t = \inf\{r \geq 0 : S_r > t\}$ is the inverse of a β-subordinator S that is independent of X.

Tools used: Mittag-Leffer functions, and the self-similarity of the β-subordinator,

$$\{S_{\lambda t}; t \geq 0\} = \{\lambda^{1/\beta} S_t; t \geq 0\} \quad \text{in distribution.}$$

Back to time fractional equation

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Fractional time Poisson equation

Let $0 < \beta < 1$. How to solve

$$\partial_t^\beta u(t,x) = \Delta u(t,x) + f(t,x)$$

with $u(0,x) = 0$?

We know from above $p(t,x,y) = \mathbb{E} p_0(E_t, x, y)$ is the fundamental solution of $\partial_t^\beta u(t,x) = \Delta u(t,x)$, where $p_0(t,x,y) = (4\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right)$. Define

$$q(t,x,y) = \partial_t^{1-\beta} p(\cdot, x, y)(t).$$

It is known in literature (Eidelman, Ivasyshen, Kouchubei, Umarov, Saydamatov, ...) that

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} q(t-s,x,y) f(s,y)\,dyds$$

solves the Poisson equation. (Duhamel's formula)

$\psi\psi\psi$

Questions

• Solution in which sense?

• Positivity: If $f(t, x, y) \geq 0$, is the solution $u(t, x) \geq 0$?

• What happens for general spatial generator $\mathcal{L}$ and for general time fractional derivatives ∂_t^w?

Classical Caputo fractional derivative

$$\frac{\partial^\beta g(t)}{\partial t^\beta} = \frac{d}{dt}\int_0^t \frac{1}{\Gamma(1-\beta)}(t-s)^{-\beta}\left((g(s)-g(0)\right)ds$$

$$= \int_0^t \frac{1}{\Gamma(1-\beta)}(t-s)^{-\beta} g'(s)ds \quad \text{if } g \text{ is Lipschitz,}$$

where $\Gamma(\lambda) = \int_0^\infty t^{\lambda-1}e^{-t}dt$.

Connection to β-stable subordinator: S_t has no drift (i.e. $\kappa = 0$) and its Lévy measure is $\nu(dx) = \frac{\beta}{\Gamma(1-\beta)}x^{-(1+\beta)}dx$. The tail measure of ν give

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General time-fractional derivative

In applications and numerical approximations, there is a need to consider more general fractional-time derivatives, for example where its value at time t may depend only on the finite range of the past from $t-\delta$ to t such as

$$\frac{d}{dt}\int_{(t-\delta)^+}^{t}(t-s)^{-\beta}\,(f(s)-f(0))\,ds.$$

Given a decreasing function w on $(0,\infty)$ with $\lim_{x\to\infty} w(x)=0$, define

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Questions

(i) Existence and uniqueness for solution of

$$(\kappa \partial_t + \partial_t^w)\, u = \mathcal{L} u \quad \text{with } u(0,x) = f(x),$$

and its probabilistic representation.

(ii) Given a strong Markov process X and subordinator S, what equation does $u(t,x) = \mathbb{E}_x \left[f(X_{E_t}) \right]$ satisfy? Here

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Subordinator

Given a constant $\kappa \geq 0$ and an unbounded right continuous non-increasing function $w(x)$ on $(0, \infty)$ with $\lim_{x \to \infty} w(x) = 0$ and $\int_0^\infty (1 \wedge x)(-dw(x)) < \infty$, there is a unique subordinator $\{S_t; t \geq 0\}$ with Laplace exponent

$$\phi(\lambda) = \kappa\lambda + \int_0^\infty (1 - e^{-\lambda x})(-dw(x)).$$

Laplace exponent: $\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}$.

Conversely, given a subordinator $\{S_t; t \geq 0\}$, there is a unique constant $\kappa \geq 0$ and a Lévy measure ν on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge x)\nu(dx) < \infty$ so that its Laplace exponent is given by above the display with $w(x) = \nu(x, \infty)$.

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From now on, we assume S_t is a subordinator with infinite Lévy measure ν and possible drift $\kappa \geq 0$. Define $w(x) = \nu(x, \infty)$.

Facts: Since $\nu(0, \infty) = \infty$, $t \mapsto S_t$ is strictly increasing. Hence the inverse subordinator E_t is continuous in t.

Suppose that $\{T_t; t \geq 0\}$ is a strongly continuous semigroup with infinitesimal generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ in some Banach space $(\mathbb{B}, \|\cdot\|)$ with the property that $\sup_{t>0} \|T_t\| < \infty$. Here $\|T_t\|$ denotes the operator norm of the linear map $T_t : \mathbb{B} \to \mathbb{B}$.

E.g. $(\mathbb{B}, \|\cdot\|) = L^p(E; \mu)$ for $p \geq 1$ or $(C_\infty(E), \|\cdot\|_\infty)$.

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Main result

Theorem (C. 2017)

For every $f \in \mathcal{D}(\mathcal{L})$, $u(t,x) := \mathbb{E}\left[T_{E_t} f(x)\right]$ is the unique solution in $(\mathbb{B}, \|\cdot\|)$ to

$$(\kappa\partial_t + \partial_t^w)\, u(t,x) = \mathcal{L}u(t,x) \quad \text{with } u(0,x) = f(x).$$

Remarks

(i) The assumption that $f \in \mathcal{D}(\mathcal{L})$ in the Theorem is to ensure that all the integrals involved in the proof are absolutely convergent in the Banach space $\mathbb{B}$. This condition can be relaxed if we formulate the time fractional equation in the weak sense when the uniformly bounded strongly continuous semigroup $\{T_t; t \geq 0\}$ is symmetric in a Hilbert space $L^2(E; m)$ and so its quadratic form can be used to formulate weak solutions. This is done in [CKKW1].

(ii) Special cases or related work: Meerschaert and Scheffler (2008) and Kolokoltsov (2011)

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Fundamental solution

When the uniformly bounded strongly continuous semigroup $\{T_t; t \geq 0\}$ has an integral kernel $p_0(t, x, y)$ with respect to some measure $m(dx)$, then there is a kernel $p(t, x, y)$ so that

$$u(t, x) := \mathbb{E}\left[T_{E_t} f(x)\right] = \int_E p(t, x, y) f(y) m(dy);$$

in other words,

$$p(t, x, y) := \mathbb{E}\left[p_0(E_t, x, y)\right] = \int_0^\infty p_0(s, x, y) d_s \mathbb{P}(E_t \leq s)$$

is the fundamental solution to the time fractional equation $(\kappa \partial_t + \partial_t^w) u = \mathcal{L} u$.

In a recent work with Kim, Kumagai and Wang, two-sided estimates on $p(t, x, y)$ are obtained when $\kappa = 0$ and $\{T_t; t \geq 0\}$ is the transition semigroup of a diffusion process that satisfies two-sided Gaussian-type estimates or of a stable-like process on metric measure spaces.

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Heat kernel estimates for $\mathcal{L}$ (particular cases)

$$p_0(t,x,y) \asymp t^{-d/\alpha} F(d(x,y)/t^{1/\alpha}).$$

1) $F(r) = \exp\left(-r^{\alpha/(\alpha-1)}\right)$ for $\alpha \geq 2$: local case

• $\alpha = 2$ when $\mathcal{L} = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$ with $\lambda^{-1} I \leq (a_{ij}(x)) \leq \lambda I$ on $\mathbb{R}^d$; Aronson 1967

• $\alpha > 2$ when $\mathcal{L}$ is the Laplacian on Sierpinski gasket or carpet; Barlow-Perkins 1988, Barlow-Bass 1992. E.g. two-dimensional Sierpinski gasket, $d = \log 3/\log 2$ and $\alpha = d_w := \log 5/\log 2$.

2) $F(r) = (1+r)^{-d-\alpha}$ with $\alpha > 0$: non-local case:

• symmetric stable-like process on Alfhors d-regular space E. C.-Kumagai 2003 ($\alpha < 2$), C.-Kumagai-Wang 2018 ($\alpha < d_w$):

$$\mathcal{L}f(x) = \mathrm{p.v.} \int_E (f(y) - f(x)) \frac{c(x,y)}{|x-y|^{d+\alpha}} \mu(dy).$$

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Fundamental solution

Particular case: $S_t = \beta$-subordinator, or Caputo derivative ∂_t^β.

Define

$$H_{\leq 1}(t, d(x,y)) = \begin{cases} t^{-\beta d/\alpha}, & d < \alpha, \\ t^{-\beta} \log\left(\dfrac{2t^\beta}{d(x,y)^\alpha}\right), & d = \alpha, \\ = t^{-\beta}/d(x,y)^{d-\alpha}, & d > \alpha, \end{cases}$$

$$H^{(c)}_{\geq 1}(t, d(x,y)) = t^{-\beta d/\alpha} \exp\Big(-(d(x,y)^\alpha/t^\beta)^{1/(\alpha-\beta)}\Big),$$

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Estimates of fundamental solution

Theorem (C.-Kim-Kumagai-Wang 2018)

(i) *Suppose* $F(r) = \exp(-r^{\alpha/(\alpha-1)})$ *with* $\alpha \geq 2$. *Then*

$$p(t,x,y) \simeq H_{\leq 1}(t, d(x,y)) \quad \text{if } d(x,y) \leq t^{\beta/\alpha},$$
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Observation

When $x \neq y$, $\lim_{t\to 0} p(t,x,y) = 0$ but

$$\lim_{t\to 0} p(t,x,x) = \infty.$$

$p(t,x,y)$ is sub-exponential decay in $d(x,y)$ in the local case and polynomial decay in non-local case.

Set up

Assume that $\{S_t, \mathbb{P}; t \geq 0\}$ is a driftless subordinatorwith infinite Lévy measure ν and having bounded density $\overline{p}(r, \cdot)$ for each $r > 0$. A sufficient condition for the latter is

$$\lim_{s\to\infty} \frac{\phi(s)}{\ln(1+s)} = \lim_{s\to\infty} \frac{1}{\ln(1+s)} \int_0^\infty (1 - e^{-sx})\, \nu(dx) = \infty.$$

(Hartman and Wintner's condition.)

Suppose that $\{P_t^0;\ t \geq 0\}$ is a uniformly bounded strongly continuous semigroup in some Banach space $(\mathbb{B}, \|\cdot\|)$ and $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is its infinitesimal generator.

Poisson equation

Theorem (C.-Kim-Kumagai-Wang, 2018+)

Let $g \in \mathcal{D}(\mathcal{L})$ and $f(t,x)$ be a function defined on $(0, T_0] \times E$ so that for a.e. $t \in (0, T_0]$, $f(t,\cdot) \in \mathcal{D}(\mathcal{L})$ and

$$\mathrm{esssup}_{t\in[0,T_0]} \|f(t,\cdot)\| + \int_0^{T_0} \|\mathcal{L} f(t,\cdot)\| dt < \infty.$$

The function

$$\begin{aligned} u(t,x) &= \mathbb{E}\left[P^0_{E_t} g(x)\right] + \mathbb{E}\left[\int_0^\infty 1_{\{S_r<t\}} P^0_r f(t-S_r,\cdot)(x) dr\right] \\ &= \mathbb{E}\left[P^0_{E_t} g(x)\right] + \int_{s=0}^t \int_{r=0}^\infty P^0_r f(t-s,\cdot)(x) \overline{p}(r,s)\, dr\, ds \end{aligned}$$

is the unique (strong) solution of $\partial^w_t u = \mathcal{L} u + f(t,x)$ on $(0, T_0] \times E$ with $u(0,x) = g(x)$ in the following sense.

Poisson equation

Theorem (C.-Kim-Kumagai-Wang, 2018+)

(i) $u(t, \cdot)$ *is well defined as an element in* $\mathbb{B}$ *for each* $t \in (0, T_0]$ *such that* $\sup_{t\in(0,T_0]} \|u(t,\cdot)\| < \infty$, $t \mapsto u(t,x)$ *is continuous in* $(\mathbb{B}, \|\cdot\|)$ *and* $\lim_{t\to 0} \|u(t,\cdot) - g\| = 0$.

(ii) *For a.e.* $t \in (0, T_0]$, $u(t, \cdot) \in \mathcal{D}(\mathcal{L})$ *and* $\mathcal{L}u(t,\cdot)$ *exists in the Banach space* $\mathbb{B}$ *with* $\int_0^{T_0} \|\mathcal{L}u(t,\cdot)\| dt < \infty$.

(iii) *For every* $T \in (0, T_0]$,

$$\int_0^T w(T-t)(u(t,\cdot) - g)\, dt = \int_0^T (f(t,\cdot) + \mathcal{L}u(t,\cdot))\, dt \quad \text{in } \mathbb{B}.$$

We also have corresponding result for weak solutions.

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We also have corresponding result for weak solutions.

Another fundamental solution

Suppose that $(\mathbb{B}, \|\cdot\|) = L^p(E;\nu)$ or $C_\infty(E)$, and the semigroup $\{P_t^0; t \geq 0\}$ has an integrable kernel $p_0(t,x,y)$ with respect to some measure $\mu(dx)$ on E. Define

$$q(t,x,y) = \int_0^\infty p_0(r,x,y)\overline{p}(r,t)dr.$$

Then the unique solution in above theorem can be expressed as

$$u(t,x) = \int_E p(t,x,y)g(y)\mu(dy) + \int_0^t \int_E q(s,x,y)f(t-s,y)\mu(dy)ds.$$

(Recall $p(t,x,y) = \mathbb{E}\left[p_0(E_t,x,y)\right]$.)

Remarks

• Positivity of $q(t, x, y)$.

• Two-sided estimates of $q(t, x, y)$.

• Stability of $p(t, x, y)$ and $q(t, x, y)$.

• An analogous probabilistic representation for solutions of Poisson equation has been obtained recently by M. E. Hernández-Hernández, V. N. Kolokoltsov and L. Toniazzi (2017) and L. Toniazzi (2018) using a different approach and in restrictive settings (Feller generator $\mathcal{L}$ in space $\mathbb{R}^d$, using Mittag-Leffer functions).

A connection

Suppose S is a special Bernstein function; that is, $\lambda \mapsto \lambda/\phi(\lambda)$ is still a Bernstein function. Let S^* be the subordinator with Laplace expondent $\lambda/\phi(\lambda)$. Suppose S_t has density function $\overline{p}(r,t)$.

Theorem (C.-Kim-Kumagai-Wang, 2018+)

For a.e. $x \neq y \in E$,

$$q(t,x,y) = \partial_t^{w^*} p(\cdot,x,y)(t)$$

in the sense that for all $t > 0$,

$$\int_0^t q(s,x,y)ds = \int_0^t w^*(t-s)p(s,x,y)ds.$$

Estimate

Particular case: $S_t = \beta$-subordinator, or Caputo derivative ∂_t^β.

Define

$$\widetilde{H}_{\le 1}(t, d(x,y)) = \begin{cases} t^{\beta-1-\beta d/\alpha}, & d < 2\alpha, \\ t^{-1-\beta} \log\left(\dfrac{2t^\beta}{d(x,y)^\alpha}\right), & d = 2\alpha, \\ = t^{-1-\beta}/d(x,y)^{d-2\alpha}, & d > 2\alpha, \end{cases}$$

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(ii) *Suppose* $F(r) = (1+r)^{-d-\alpha}$. *Then,*

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Where these formula come from?

Observations: Suppose g is locally Lipschitz on $[0, \infty)$.

(i) $\partial_t^w g(t)$ exists for a.e. $t > 0$ and

$$\partial_t^w g(t) = \int_0^t w(t-s)g'(s)ds.$$

(ii) Extending $g(s) = g(0)$ for $s < 0$, then

$$\partial_t^w g(t) = \int_0^\infty (g(t-z) - g(t))\nu(dz) = \mathcal{A}^* g(t).$$

Here $\mathcal{A}^*$ is the infinitesimal generator of the Lévy process $-S_t$.

Space-time process

Key observation: $-\partial_t^w + \mathcal{L}$ is the infinitesimal generator of $(-S_t, X_t)$.

Suppose that $u(t,x)$ is a solution to $\partial_t^w u = \mathcal{L}u + f(t,x)$ on $(0, T_0] \times E$ with $u(0,x) = g(x)$. For each fixed $T \in (0, T_0]$, consider $u(T - S_t, X_t)$. Then

$$\begin{aligned} M_t &= u(T - S_t, X_t) - \int_0^t (-\partial_t^w + \mathcal{L}) u(T - S_t, X_t) dt \\ &= u(T - S_t, X_t) + \int_0^t f(T - S_t, X_t) dt \end{aligned}$$

is a martingale. So $\mathbb{E}_x M_0 = \mathbb{E}_x M_{E_T}$. That is,

$$\begin{aligned} u(T, x) &= \mathbb{E}_x g(X_{E_T}) + \mathbb{E}_x \int_0^{E_T} f(T - S_t, X_t) dt \\ &= \mathbb{E} P_{E_T} g(x) + \mathbb{E} \int_0^\infty 1_{\{S_t < T\}} P_t f(T - S_t, \cdot)(x) dt. \end{aligned}$$

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However, there is a problem!

We do not know a prior if $u(T-t, x)$ is in the domain of the generator $-\partial_t^w + \mathcal{L}$.

To rigorously prove these formulas, we use a different approach by studying the properties of subordinator and inverse subordinator. Here is an example.

Theorem (C. 2017)

There is a Borel set $\mathcal{N} \subset (0, \infty)$ having zero Lebesgue measure such that for every $t \in (0, \infty) \setminus \mathcal{N}$, the inverse subordinator E_t has a density function given by

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$$\frac{d}{dr}\mathbb{P}(E_t \le r) = \int_0^t w(t-s)\overline{p}(r,s)ds, \quad r > 0.$$

However, there is a problem!

We do not know a prior if $u(T-t,x)$ is in the domain of the generator $-\partial_t^w + \mathcal{L}$.

To rigorously prove these formulas, we use a different approach by studying the properties of subordinator and inverse subordinator. Here is an example.

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Why it is so: a plausible proof

For $a > 0$, let $f = 1_{[0,a]}$.

$$\begin{aligned}
\frac{d}{dt}\mathbb{P}(S_t \leq a) &= \frac{d}{dt}P_t^{(S)}f(0) = \mathcal{L}^{(S)}P_t^{(S)}f(0) \\
&= \int_0^\infty \left(P_t^{(S)}f(z) - P_t^{(S)}f(0)\right)\nu(dz) \\
&= \int_0^\infty \mathbb{P}(a - z < S_t \leq a)dw(z) \\
&= -\int_0^\infty w(z)d_z\mathbb{P}(a - z < S_t \leq a) \\
&= -\int_0^a w(z)\overline{p}(t, a - z)dz.
\end{aligned}$$

(Used integration by parts formula and the fact that $\lim_{z\downarrow 0} zw(z) = 0$ and $\lim_{z\to\infty} w(z) = 0$.) Then $\mathbb{P}(E_s \leq r) = \mathbb{P}(S_r > s)$ gives the formula.

But ...

The indicator function $f = 1_{[0,a]}$ is not in the domain of the Feller generator of the subordinator. Thus needs a different proof. Take Laplace transform on both sides to verify.

Examples

(i) When $\{S_t; t \geq 0\}$ is a β-subordinator with $0 < \beta < 1$ with Laplace exponent $\phi(\lambda) = \lambda^\beta$, Then S_t has no drift (i.e. $\kappa = 0$) and its Lévy measure is $\mu(dx) = \frac{\beta}{\Gamma(1-\beta)} x^{-(1+\beta)} dx$. Hence

$$w(x) := \mu(x, \infty) = \int_x^\infty \frac{\beta}{\Gamma(1-\beta)} y^{-(1+\beta)} dy = \frac{x^{-\beta}}{\Gamma(1-\beta)}.$$

Thus the time fractional derivative $\partial_t^w f$ is exactly the Caputo derivative of order β. In this case, our Theorem recovers the main result of Baeumer-Meerschaert (2001) and Meerschaert-Scheffler (2004).

Truncated stable-subordinator

(ii) A truncated β-stable subordinator $\{S_t; t \geq 0\}$ is driftless and has Lévy measure

$$\mu_\delta(dx) = \frac{\beta}{\Gamma(1-\beta)} x^{-(1+\beta)} 1_{(0,\delta]}(x) dx$$

for some $\delta > 0$. In this case,

$$\begin{aligned} w_\delta(x) &:= \mu_\delta(x,\infty) = 1_{\{0<x\leq\delta\}} \int_x^\delta \frac{\beta}{\Gamma(1-\beta)} y^{-(1+\beta)} dy \\ &= \frac{1}{\Gamma(1-\beta)} \left(x^{-\beta} - \delta^{-\beta} \right) 1_{(0,\delta]}(x). \end{aligned}$$

The corresponding the fractional derivative is

$$\partial_t^{w_\delta} f(t) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_{(t-\delta)^+}^t \left((t-s)^{-\beta} - \delta^{-\beta} \right) (f(s) - f(0)) ds.$$

Stability

Clearly, as $\lim_{\delta\to\infty} w_\delta(x) = w(x) := \frac{1}{\Gamma(1-\beta)} x^{-\beta}$. Consequently, $\partial_t^{w_\delta} f(t) \to \partial_t^{w} f(t)$, the Caputo derivative of f of order β, in the distributional sense as $\delta \to 0$. Using the probabilistic representation in the main Theorem, one can deduce that as $\delta \to \infty$, the solution to the equation $\partial_t^{w_\delta} u = \mathcal{L} u$ with $u(0,x) = f(x)$ converges to the solution of $\partial_t^{\beta} u = \mathcal{L} u$ with $u(0,x) = f(x)$.

(iii) If we define

$$\eta_\delta(r) = \frac{\Gamma(2-\beta)\,\delta^{\beta-1}}{\beta} w_\delta(r) = (1-\beta)\delta^{\beta-1}\left(x^{-\beta} - \delta^{-\beta}\right) 1_{(0,\delta]}(x),$$

then $\eta_\delta(r)$ converges weakly to the Dirac measure concentrated at 0 as $\delta \to 0$. So $\partial_t^{\eta_\delta} f(t)$ converges to $f'(t)$ for every differentiable f. It can be shown that the subordinator corresponding to η_δ, that is, subordinator with Lévy measure

$$\nu_\delta(dx) := \frac{(1-\beta)\,\delta^{\beta-1}}{\beta} x^{-(1+\beta)} 1_{(0,\delta]}(x) dx,$$

converges as $\delta \to 0$ to deterministic motion t moving at constant speed 1. Using the main Theorem, one can show that the solution to the equation $\partial_t^{\eta_\delta} u(t,x) = \mathcal{L}u(t,x)$ with $u(0,x) = f(x)$ converges to the solution of the heat equation $\partial_t u = \mathcal{L}u$ with $u(0,x) = f(x)$.

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Thank you!