Fundamental Solutions to Time fractional Poisson equations

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Given a Markov process \((X, \mathbb{P}_x, x \in E)\) on \(E\), its transition semigroup \(\{P_t; t \geq 0\}\) is given by

\[ P_t\phi(x) = \mathbb{E}_x[\phi(X_t)]. \]

The infinitesimal generator \(\mathcal{L}\) of \(X\) is

\[ \mathcal{L}\phi(x) = \lim_{t \to 0} \frac{P_t\phi(x) - \phi(x)}{t}. \]

Hence \(u(t, x) = P_t\phi(x)\) solves \(\frac{\partial u}{\partial t} = \mathcal{L}u\) with \(u(0, x) = \phi(x)\).

- When \(X\) is Brownian motion on \(\mathbb{R}^d\), \(\mathcal{L} = \frac{1}{2} \Delta\).
- When \(X\) is an absorbing (or reflecting) Brownian motion in \(D \subset \mathbb{R}^d\), \(\mathcal{L}\) is the Dirichlet (or Neumann Laplacian in \(D\).
- When \(X\) is a rotationally symmetric \(\alpha\)-stable process, \(\mathcal{L} = -(-\Delta)^{\alpha/2}\).
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Generator

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Note that \( P_t\phi(x) - \phi(x) = \int_0^t \mathcal{L} P_s \phi(x) dt \). If \( \phi \in \text{Dom}(\mathcal{L}) \), then

\[
P_t \phi(x) - \phi(x) = \int_0^t P_s \mathcal{L} \phi(x) dt.
\]

In other words, \( \mathbb{E}_x M_t = 0 \) for \( t \geq 0 \) and \( x \in E \), where

\[
M_t = \phi(X_t) - \phi(X_0) - \int_0^t \mathcal{L} \phi(X_s) ds.
\]

This implies that \( M_t \) is a martingale.

Define \( \tau_D = \inf\{ t \geq 0 : X_t \notin D \} \). Then (under suitable condition) \( \mathbb{E}_x M_{\tau_D} = 0 \). So if \( \mathcal{L} \phi = 0 \) in \( D \) with \( \phi(x) = g(x) \) on \( D^c \), then

\[
\phi(x) = \mathbb{E}_x [g(X_{\tau_D})], \quad x \in D.
\]

S. Kakutani (1944): used Brownian motion to solve classical Dirichlet boundary value problem.
Under suitable conditions, the solution to
\[ \frac{\partial u}{\partial t} = \mathcal{L}u + f(t, x) \quad \text{with } u(0, x) = \phi(x) \]
is given by
\[
\begin{align*}
  u(t, x) & = P_t \phi(x) + \int_0^t P_{t-s} f(s, \cdot)(x) ds \\
            & = \mathbb{E}_x \phi(X_t) + \mathbb{E}_x \int_0^t f(t - s, X_s) ds.
\end{align*}
\]
Why? Formally,
\[
\begin{align*}
  \frac{\partial u}{\partial t} & = \frac{\partial}{\partial t} P_t \phi(x) + P_0 f(t, \cdot)(x) + \int_0^t \frac{\partial}{\partial t} P_{t-s} f(s, \cdot)(x) ds \\
                             & = \mathcal{L} P_t \phi(x) + f(t, x) + \int_0^t \mathcal{L} P_{t-s} f(s, \cdot)(x) ds \\
                             & = \mathcal{L} u(t, \cdot)(x) + f(t, x).
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    &= \mathcal{L}P_t\phi(x) + f(t, x) + \int_0^t \mathcal{L}P_{t-s}f(s, \cdot)(x)ds \\
    &= \mathcal{L}u(t, \cdot)(x) + f(t, x).
\end{align*}
\]
The goal of this talk is to study

\[ \partial_t^w \nu = \mathcal{L} \nu + f(t, x), \]

where

\[ \partial_t^w g(t) := \frac{d}{dt} \int_0^t w(t - s)(g(s) - g(0))ds. \]

Here \( w \geq 0 \) is a decreasing function with \( w(0) = \infty \) and
\( \int_0^\infty (t \wedge 1)(-dw(t)) < \infty. \)

When \( w(r) = \frac{1}{\Gamma(1-\beta)} r^{-\beta} \) with \( 0 < \beta < 1 \), \( \partial_t^w g(t) \) is the classical Caputo fractional derivative \( \partial_t^\beta \) of order \( \beta \).
Why do we care

- Non-local spatial operators can be used to model anomalous superdiffusion that describe particles move faster than Brownian motion (e.g. the random walker remains in motion without changing direction for a time that follows a Pareto-Lévy distribution).

- Fractional time equation has a close connection to anomalous subdiffusions that describe particles move slower than Brownian motion (or the original underlying spatial motion), for example, due to particle sticking and trapping.

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Subordinate Markov Process

Suppose $S = \{S_t; t \geq 0\}$ is a subordinator independent of $X$ with Laplace exponent $\phi$:

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}.$$  

There is a unique $\kappa \geq 0$ and a measure $\nu(dx)$ with $\int_0^{\infty} (1 \wedge x) \nu(dx) < \infty$ so that

$$\phi(\lambda) = \kappa \lambda + \int_0^{\infty} (1 - e^{-\lambda x}) \nu(dx).$$

$X_{S_t}$ is a Markov process, called subordinate Markov process. When $X$ is symmetric, the infinitesimal generator of $X_{S_t}$ is $\mathcal{L}_\phi := -\phi(-\mathcal{L}).$

Hence $u(t, x) := \mathbb{E}_x[f(X_{S_t})]$ solves $\frac{\partial u}{\partial t} = \mathcal{L}_\phi u$ with $u(0, x) = f(x).$
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Example: stable subordinator

When \( \{S_t; t \geq 0\} \) is a \( \beta \)-subordinator with \( 0 < \beta < 1 \) with Laplace exponent \( \phi(\lambda) = \lambda^\beta \), the infinitesimal generator of \( X_{S_t} \) is \(-(-\mathcal{L})^\beta\).

When \( X \) is Brownian motion in \( \mathbb{R}^d \), \( X_{S_t} \) is a rotationally symmetric \((2\beta)\)-stable process in \( \mathbb{R}^d \), whose infinitesimal generator is \(-(-\Delta)^\beta =: \Delta^\beta\). It can also be expressed as

\[
\Delta^\beta f(x) = \lim_{\varepsilon \to 0} \int_{\{|y-x| > \varepsilon\}} (f(y) - f(x)) \frac{c(d, \alpha)}{|y - x|^{d+2\beta}} dy
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= \int_{\mathbb{R}^d} (f(x + z) - f(x) - \nabla f(x) \cdot z \mathbf{1}_{|z| \leq 1}) \frac{c(d, \alpha)}{|z|^{d+2\beta}} dz.
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Space dependent non-local operator: for fundamental solutions

- Non-symmetric: C.-Zhang 2016, 2018, · · ·
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- Symmetric case: C.-Kumagai 2003, 2008, 2010, ...
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Theorem (Baeumer-Meerschaert, 2001; Meerschaert-Scheffler, 2004): $u(t, x) = \mathbb{E}_x[f(X_{E_t})]$ solves

$$\frac{\partial^\beta u(t, x)}{\partial t^\beta} = \mathcal{L}_x u(t, x), \quad u(0, x) = f(x).$$

Here $E_t = \inf\{r \geq 0 : S_r > t\}$ is the inverse of a $\beta$-subordinator $S$ that is independent of $X$.

Tools used: Mittag-Leffler functions, and the self-similarity of the $\beta$-subordinator,

$$\{S_{\lambda t}; t \geq 0\} = \{\lambda^{1/\beta} S_t; t \geq 0\} \text{ in distribution.}$$
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$$\{S_{\lambda t}; t \geq 0\} = \{\lambda^{1/\beta} S_t; t \geq 0\} \text{ in distribution.}$$
Let $0 < \beta < 1$. How to solve

$$\partial_t^\beta u(t, x) = \Delta u(t, x) + f(t, x)$$

with $u(0, x) = 0$?

We know from above $p(t, x, y) = \mathbb{E}p_0(E_t, x, y)$ is the fundamental solution of $\partial_t^\beta u(t, x) = \Delta u(t, x)$, where $p_0(t, x, y) = (4\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right)$. Define

$$q(t, x, y) = \partial_t^{1-\beta} p(\cdot, x, y)(t).$$

It is known in literature (Eidelman, Ivasyshen, Kouchubei, Umarov, Saydamatov, ...) that

$$u(t, x) = \int_0^t \int_{\mathbb{R}^d} q(t - s, x, y)f(s, y)dyds$$

solves the Poisson equation. (Duhamel’s formula)
Questions

• Solution in which sense?

• Positivity: If $f(t, x, y) \geq 0$, is the solution $u(t, x) \geq 0$?

• What happens for general spatial generator $\mathcal{L}$ and for general time fractional derivatives $\partial_t^w$?
Classical Caputo fractional derivative

\[
\frac{\partial^\beta g(t)}{\partial t^\beta} = \frac{d}{dt} \int_0^t \frac{1}{\Gamma(1 - \beta)} (t - s)^{-\beta} ((g(s) - g(0)) \, ds
\]

\[
= \int_0^t \frac{1}{\Gamma(1 - \beta)} (t - s)^{-\beta} g'(s) \, ds \quad \text{if } g \text{ is Lipschitz},
\]

where \( \Gamma(\lambda) = \int_0^\infty t^{\lambda-1} e^{-t} \, dt \).

Connection to \( \beta \)-stable subordinator: \( S_t \) has no drift (i.e. \( \kappa = 0 \)) and its Lévy measure is \( \nu(dx) = \frac{\beta}{\Gamma(1-\beta)} x^{-(1+\beta)} \, dx \). The tail measure of \( \nu \) give

\[
w(x) := \nu(x, \infty) = \int_x^\infty \frac{\beta}{\Gamma(1-\beta)} y^{-(1+\beta)} \, dy = \frac{x^{-\beta}}{\Gamma(1 - \beta)}.
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In applications and numerical approximations, there is a need to consider more general fractional-time derivatives, for example where its value at time $t$ may depend only on the finite range of the past from $t - \delta$ to $t$ such as

$$\frac{d}{dt} \int_{(t-\delta)^{+}}^{t} (t - s)^{-\beta} (f(s) - f(0)) \, ds.$$ 

Given a decreasing function $w$ on $(0, \infty)$ with $\lim_{x \to \infty} w(x) = 0$, define

$$\partial_t^w f(t) = \frac{d}{dt} \int_{0}^{t} w(t - s) (f(s) - f(0)) \, ds,$$
General time-fractional derivative

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Given a decreasing function $w$ on $(0, \infty)$ with $\lim_{x \to \infty} w(x) = 0$, define

$$\partial^w_t f(t) = \frac{d}{dt} \int_{0}^{t} w(t-s) (f(s) - f(0)) \, ds,$$
(i) Existence and uniqueness for solution of

\[( \kappa \partial_t + \partial_t^w ) u = \mathcal{L} u \quad \text{with} \quad u(0, x) = f(x), \]

and its probabilistic representation.

(ii) Given a strong Markov process \( X \) and subordinator \( S \), what equation does \( u(t, x) = \mathbb{E}_x [f(X_{E_t})] \) satisfy? Here

\[ E_t = \inf\{s > 0 : S_s > t\}. \]
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\[E_t = \inf\{s > 0 : S_s > t\}.\]
Given a constant $\kappa \geq 0$ and an unbounded right continuous non-increasing function $w(x)$ on $(0, \infty)$ with $\lim_{x \to \infty} w(x) = 0$ and $\int_0^\infty (1 \wedge x)(-dw(x)) < \infty$, there is a unique subordinator $\{S_t; t \geq 0\}$ with Laplace exponent

$$\phi(\lambda) = \kappa \lambda + \int_0^\infty (1 - e^{-\lambda x})(-dw(x)).$$

Laplace exponent: $\mathbb{E} \left[ e^{-\lambda S_t} \right] = e^{-t\phi(\lambda)}$.

Conversely, given a subordinator $\{S_t; t \geq 0\}$, there is a unique constant $\kappa \geq 0$ and a Lévy measure $\nu$ on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge x)\nu(dx) < \infty$ so that its Laplace exponent is given by above the display with $w(x) = \nu(x, \infty)$.
Given a constant $\kappa \geq 0$ and an unbounded right continuous non-increasing function $w(x)$ on $(0, \infty)$ with $\lim_{x \to \infty} w(x) = 0$ and $\int_0^\infty (1 \wedge x)(-dw(x)) < \infty$, there is a unique subordinator \(\{S_t; t \geq 0\}\) with Laplace exponent

$$\phi(\lambda) = \kappa \lambda + \int_0^\infty (1 - e^{-\lambda x})(-dw(x)).$$

**Laplace exponent:** $\mathbb{E} \left[ e^{-\lambda S_t} \right] = e^{-t\phi(\lambda)}$.

Conversely, given a subordinator \(\{S_t; t \geq 0\}\), there is a unique constant $\kappa \geq 0$ and a Lévy measure $\nu$ on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge x)\nu(dx) < \infty$ so that its Laplace exponent is given by above the display with $w(x) = \nu(x, \infty)$. 
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Fundamental Solutions to Time fractional Poisson equations
From now on, we assume $S_t$ is a subordinator with infinite Lévy measure $\nu$ and possible drift $\kappa \geq 0$. Define $w(x) = \nu(x, \infty)$.

**Facts:** Since $\nu(0, \infty) = \infty$, $t \mapsto S_t$ is strictly increasing. Hence the inverse subordinator $E_t$ is continuous in $t$.

Suppose that $\{T_t; t \geq 0\}$ is a strongly continuous semigroup with infinitesimal generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ in some Banach space $(\mathbb{B}, \| \cdot \|)$ with the property that $\sup_{t>0} \| T_t \| < \infty$. Here $\| T_t \|$ denotes the operator norm of the linear map $T_t : \mathbb{B} \rightarrow \mathbb{B}$.

E.g. $(\mathbb{B}, \| \cdot \|) = L^p(E; \mu)$ for $p \geq 1$ or $(C_\infty(E), \| \cdot \|_\infty)$.
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---

**Framework**

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Foundamental Solutions to Time fractional Poisson equations
Main result

Theorem (C. 2017)

For every \( f \in \mathcal{D}(\mathcal{L}) \), \( u(t, x) := \mathbb{E}[T_{E_t}f(x)] \) is the unique solution in \((\mathcal{B}, \| \cdot \|)\) to

\[
(\kappa \partial_t + \partial_t^w) u(t, x) = \mathcal{L}u(t, x) \quad \text{with} \quad u(0, x) = f(x).
\]
(i) The assumption that $f \in D(\mathcal{L})$ in the Theorem is to ensure that all the integrals involved in the proof are absolutely convergent in the Banach space $\mathbb{B}$. This condition can be relaxed if we formulate the time fractional equation in the weak sense when the uniformly bounded strongly continuous semigroup $\{T_t; t \geq 0\}$ is symmetric in a Hilbert space $L^2(E; m)$ and so its quadratic form can be used to formulate weak solutions. This is done in [CKKW1].

(ii) Special cases or related work: Meerschaert and Scheffler (2008) and Kolokoltsov (2011)

(iii) There are very limited results on uniqueness.

(iv) One needs to be very careful when dealing with time frictional equations due to nature of singular integrals. Probabilistic representation turns out to be quite effective to overcome these difficulties.
Remarks

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Remarks

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(iv) One needs to be very careful when dealing with time frictional equations due to nature of singular integrals. Probabilistic representation turns out to be quite effective to overcome these difficulties.
When the uniformly bounded strongly continuous semigroup \( \{ T_t; t \geq 0 \} \) has an integral kernel \( p_0(t, x, y) \) with respect to some measure \( m(dx) \), then there is a kernel \( p(t, x, y) \) so that

\[
    u(t, x) := \mathbb{E} [T_t f(x)] = \int_{E} p(t, x, y) f(y) m(dy);
\]

in other words,

\[
    p(t, x, y) := \mathbb{E} [p_0(E_t, x, y)] = \int_0^\infty p_0(s, x, y) d_s \mathbb{P}(E_t \leq s)
\]

is the fundamental solution to the time fractional equation

\[(\kappa \partial_t + \partial_t^w) u = \mathcal{L} u.\]

In a recent work with Kim, Kumagai and Wang, two-sided estimates on \( p(t, x, y) \) are obtained when \( \kappa = 0 \) and \( \{ T_t; t \geq 0 \} \) is the transition semigroup of a diffusion process that satisfies two-sided Gaussian-type estimates or of a stable-like process on metric measure spaces.

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Fundamental Solutions to Time fractional Poisson equations
Fundamental solution

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Heat kernel estimates for $\mathcal{L}$ (particular cases)

$$p_0(t, x, y) \asymp t^{-d/\alpha} F(d(x, y)/t^{1/\alpha}).$$

1) $F(r) = \exp \left( -r^\alpha/(\alpha-1) \right)$ for $\alpha \geq 2$: local case
   - $\alpha = 2$ when $\mathcal{L} = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right)$ with $\lambda^{-1} I \leq (a_{ij}(x)) \leq \lambda I$ on $\mathbb{R}^d$; Aronson 1967
   - $\alpha > 2$ when $\mathcal{L}$ is the Laplacian on Sierpinski gasket or carpet; Barlow-Perkins 1988, Barlow-Bass 1992. E.g. two-dimensional Sierpinski gasket, $d = \log 3/\log 2$ and $\alpha = d_w := \log 5/\log 2$.

2) $F(r) = (1 + r)^{-d-\alpha}$ with $\alpha > 0$: non-local case:
   - symmetric stable-like process on Alfhors $d$-regular space $E$. C.-Kumagai 2003 ($\alpha < 2$), C.-Kumagai-Wang 2018 ($\alpha < d_w$):

$$\mathcal{L} f(x) = \text{p.v.} \int_E (f(y) - f(x)) \frac{c(x, y)}{|x - y|^{d+\alpha}} \mu(dy).$$
Heat kernel estimates for $\mathcal{L}$ (particular cases)

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Particular case: $S_t = \beta$-subordinator, or Caputo derivative $\partial_t^\beta$.

Define

$$H_{\leq 1}(t, d(x, y)) = \begin{cases} t^{-\beta d/\alpha}, & d < \alpha, \\ t^{-\beta} \log \left( \frac{2t^\beta}{d(x, y)^\alpha} \right), & d = \alpha, \\ t^{-\beta} / d(x, y)^{d-\alpha}, & d > \alpha, \end{cases}$$

$$H_{\geq 1}^{(c)}(t, d(x, y)) = t^{-\beta d/\alpha} \exp \left( - \left( d(x, y)^\alpha / t^\beta \right)^{1/(\alpha-\beta)} \right),$$

$$H_{\geq 1}^{(i)}(t, d(x, y)) = t^\beta / d(x, y)^{d+\alpha}.$$
Theorem (C.-Kim-Kumagai-Wang 2018)

(i) Suppose $F(r) = \exp(-r^{\alpha}/(\alpha - 1))$ with $\alpha \geq 2$. Then

\[
p(t, x, y) \asymp H_{\leq 1}(t, d(x, y)) \quad \text{if } d(x, y) \leq t^{\beta/\alpha},
\]
\[
p(t, x, y) \asymp H_{\geq 1}^{(c)}(t, d(x, y)) \quad \text{if } d(x, y) \geq t^{\beta/\alpha}.
\]

(ii) Suppose $F(r) = (1 + r)^{-d - \alpha}$. Then,

\[
p(t, x, y) \asymp H_{\leq 1}(t, d(x, y)) \quad \text{if } d(x, y) \leq t^{\beta/\alpha},
\]
\[
p(t, x, y) \asymp H_{\geq 1}^{(j)}(t, d(x, y)) \quad \text{if } d(x, y) \geq t^{-\beta/\alpha}.
\]

\[
H_{\leq 1}(t, d(x, y)) = \begin{cases} 
t^{-\beta d/\alpha}, & d < \alpha, \\
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\end{cases}
\]

\[
H_{\geq 1}^{(c)}(t, d(x, y)) = t^{-\beta d/\alpha} \exp \left(- (d(x, y)^\alpha / t^\beta)^{1/(\alpha-\beta)}\right), \quad H_{\geq 1}^{(j)}(t, d(x, y)) = t^\beta / d(x, y)^{d+\alpha}.
\]
When $x \neq y$, $\lim_{t \to 0} p(t, x, y) = 0$ but

$$\lim_{t \to 0} p(t, x, x) = \infty.$$  

$p(t, x, y)$ is sub-exponential decay in $d(x, y)$ in the local case and polynomial decay in non-local case.
Set up

Assume that \( \{ S_t, P; t \geq 0 \} \) is a driftless subordinator with infinite Lévy measure \( \nu \) and having bounded density \( \bar{p}(r, \cdot) \) for each \( r > 0 \). A sufficient condition for the latter is

\[
\lim_{s \to \infty} \frac{\phi(s)}{\ln(1 + s)} = \lim_{s \to \infty} \frac{1}{\ln(1 + s)} \int_0^\infty (1 - e^{-sx}) \nu(dx) = \infty.
\]

(Hartman and Wintner’s condition.)

Suppose that \( \{ P^0_t; t \geq 0 \} \) is a uniformly bounded strongly continuous semigroup in some Banach space \((B, \| \cdot \|)\) and \((L, D(L))\) is its infinitesimal generator.
Theorem (C.-Kim-Kumagai-Wang, 2018+)

Let \( g \in D(\mathcal{L}) \) and \( f(t, x) \) be a function defined on \((0, T_0] \times E\) so that for a.e. \( t \in (0, T_0] \), \( f(t, \cdot) \in D(\mathcal{L}) \) and

\[
\text{esssup}_{t \in [0, T_0]} \| f(t, \cdot) \| + \int_0^{T_0} \| \mathcal{L} f(t, \cdot) \| \, dt < \infty.
\]

The function

\[
u(t, x) = \mathbb{E} \left[ P^0_{E_t} g(x) \right] + \mathbb{E} \left[ \int_0^\infty \mathbb{1}_{\{S_r < t\}} P^0_r f(t - S_r, \cdot)(x) \, dr \right]
\]

\[
= \mathbb{E} \left[ P^0_{E_t} g(x) \right] + \int_0^t \int_r^\infty P^0_r f(t - s, \cdot)(x) \overline{p}(r, s) \, dr \, ds
\]

is the unique (strong) solution of \( \partial^w_t u = \mathcal{L} u + f(t, x) \) on \((0, T_0] \times E\) with \( u(0, x) = g(x) \) in the following sense.
Theorem (C.-Kim-Kumagai-Wang, 2018+)

(i) $u(t, \cdot)$ is well defined as an element in $\mathbb{B}$ for each $t \in (0, T_0]$ such that $\sup_{t \in (0, T_0]} \| u(t, \cdot) \| < \infty$, $t \mapsto u(t, x)$ is continuous in $(\mathbb{B}, \| \cdot \|)$ and $\lim_{t \to 0} \| u(t, \cdot) - g \| = 0$.

(ii) For a.e. $t \in (0, T_0]$, $u(t, \cdot) \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}u(t, \cdot)$ exists in the Banach space $\mathbb{B}$ with $\int_0^{T_0} \| \mathcal{L}u(t, \cdot) \| dt < \infty$.

(iii) For every $T \in (0, T_0]$,

$$
\int_0^T w(T - t)(u(t, \cdot) - g) \, dt = \int_0^T (f(t, \cdot) + \mathcal{L}u(t, \cdot)) \, dt \quad \text{in} \ \mathbb{B}.
$$

We also have corresponding result for weak solutions.
Theorem (C.-Kim-Kumagai-Wang, 2018+)

(i) $u(t, \cdot)$ is well defined as an element in $\mathbb{B}$ for each $t \in (0, T_0]$ such that $\sup_{t \in (0, T_0]} \|u(t, \cdot)\| < \infty$, $t \mapsto u(t, x)$ is continuous in $(\mathbb{B}, \|\cdot\|)$ and $\lim_{t \to 0} \|u(t, \cdot) - g\| = 0$.

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$$\int_0^T w(T - t)(u(t, \cdot) - g) \, dt = \int_0^T (f(t, \cdot) + \mathcal{L}u(t, \cdot)) \, dt \quad \text{in } \mathbb{B}.$$ 

We also have corresponding result for weak solutions.
Another fundamental solution

Suppose that \((\mathcal{B}, \| \cdot \|) = L^p(E; \nu)\) or \(C_{\infty}(E)\), and the semigroup \(\{P^0_t; t \geq 0\}\) has an integrable kernel \(p_0(t, x, y)\) with respect to some measure \(\mu(dx)\) on \(E\). Define

\[
q(t, x, y) = \int_0^\infty p_0(r, x, y)\overline{p}(r, t)dr.
\]

Then the unique solution in above theorem can be expressed as

\[
u(t, x) = \int_E p(t, x, y)g(y)\mu(dy) + \int_0^t \int_E q(s, x, y)f(t-s, y)\mu(dy)ds.
\]

(Recall \(p(t, x, y) = \mathbb{E}[p_0(E_t, x, y)]\).)
Remarks

• Positivity of $q(t, x, y)$.

• Two-sided estimates of $q(t, x, y)$.

• Stability of $p(t, x, y)$ and $q(t, x, y)$.

• An analogous probabilistic representation for solutions of Poisson equation has been obtained recently by M. E. Hernández-Hernández, V. N. Kolokoltsov and L. Toniazzi (2017) and L. Toniazzi (2018) using a different approach and in restrictive settings (Feller generator $\mathcal{L}$ in space $\mathbb{R}^d$, using Mittag-Leffer functions).
Suppose $S$ is a special Bernstein function; that is, $\lambda \mapsto \lambda / \phi(\lambda)$ is still a Bernstein function. Let $S^*$ be the subordinator with Laplace exponent $\lambda / \phi(\lambda)$. Suppose $S_t$ has density function $\overline{\rho}(r, t)$.

**Theorem (C.-Kim-Kumagai-Wang, 2018+)**

For a.e. $x \neq y \in E$,

$$q(t, x, y) = \partial_t^{w^*} p(\cdot, x, y)(t)$$

in the sense that for all $t > 0$,

$$\int_0^t q(s, x, y)ds = \int_0^t w^*(t - s)p(s, x, y)ds.$$
Particular case: $S_t = \beta$-subordinator, or Caputo derivative $\partial_t^\beta$.

Define

$$\tilde{H}_{\leq 1}(t, d(x, y)) = \begin{cases} t^{\beta-1-\beta d/\alpha}, & d < 2\alpha, \\ t^{-1-\beta} \log \left( \frac{2t^\beta}{d(x, y)^\alpha} \right), & d = 2\alpha, \\ = t^{-1-\beta} / d(x, y)^{d-2\alpha}, & d > 2\alpha, \end{cases}$$

$$\tilde{H}^{(c)}_{\geq 1}(t, d(x, y)) = t^{\beta-1-\beta d/\alpha} \exp \left( - (d(x, y)^\alpha / t^\beta)^{1/(\alpha-\beta)} \right),$$

$$\tilde{H}^{(j)}_{\geq 1}(t, d(x, y)) = t^{2\beta-1} / d(x, y)^{d+\alpha}.$$
Estimates of fundamental solution

Theorem (C.-Kim-Kumagai-Wang 2018+)

(i) Suppose \( F(r) = \exp(-r^{\alpha}/(\alpha-1)) \) with \( \alpha \geq 2 \). Then

\[
q(t, x, y) \approx \tilde{H}_{\leq 1}(t, d(x, y)) \quad \text{if } d(x, y) \leq t^{\beta/\alpha}, \\
q(t, x, y) \approx \tilde{H}^{(c)}_{\geq 1}(t, d(x, y)) \quad \text{if } d(x, y) \geq t^{\beta/\alpha}.
\]

(ii) Suppose \( F(r) = (1 + r)^{-d-\alpha} \). Then,

\[
q(t, x, y) \approx \tilde{H}_{\leq 1}(t, d(x, y)) \quad \text{if } d(x, y) \leq t^{\beta/\alpha}, \\
q(t, x, y) \approx \tilde{H}^{(j)}_{\geq 1}(t, d(x, y)) \quad \text{if } d(x, y) \geq t^{\beta/\alpha}.
\]
Observations: Suppose $g$ is locally Lipschitz on $[0, \infty)$. 

(i) $\partial_t^w g(t)$ exists for a.e. $t > 0$ and

$$\partial_t^w g(t) = \int_0^t w(t - s)g'(s)ds.$$ 

(ii) Extending $g(s) = g(0)$ for $s < 0$, then

$$\partial_t^w g(t) = \int_0^\infty (g(t - z) - g(t))\nu(dz) = \mathcal{A}^* g(t).$$

Here $\mathcal{A}^*$ is the infinitesimal generator of the Lévy process $-S_t$. 
Key observation: $-\partial_t^w + \mathcal{L}$ is the infinitesimal generator of $(-S_t, X_t)$.

Suppose that $u(t, x)$ is a solution to $\partial_t^w u = \mathcal{L} u + f(t, x)$ on $(0, T_0] \times E$ with $u(0, x) = g(x)$. For each fixed $T \in (0, T_0]$, consider $u(T - S_t, X_t)$. Then

$$M_t = u(T - S_t, X_t) - \int_0^t (-\partial_t^w + \mathcal{L}) u(T - S_t, X_t) dt$$

$$= u(T - S_t, X_t) + \int_0^t f(T - S_t, X_t) dt$$

is a martingale. So $\mathbb{E}_x M_0 = \mathbb{E}_x M_{E_T}$. That is,

$$u(T, x) = \mathbb{E}_x g(X_{E_T}) + \mathbb{E}_x \int_0^{E_T} f(T - S_t, X_t) dt$$

$$= \mathbb{E} P_{E_T} g(x) + \mathbb{E} \int_0^{\infty} \mathbf{1}_{\{S_t < T\}} P_t f(T - S_t, \cdot)(x) dt.$$
Key observation: $-\partial_t^w + \mathcal{L}$ is the infinitesimal generator of $(-S_t, X_t)$.

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However, there is a problem!

We do not know a prior if \( u(T-t,x) \) is in the domain of the generator \(-\partial_t^w + \mathcal{L}\).

To rigorously prove these formulas, we use a different approach by studying the properties of subordinator and inverse subordinator. Here is an example.

**Theorem (C. 2017)**

There is a Borel set \( \mathcal{N} \subset (0, \infty) \) having zero Lebesgue measure such that for every \( t \in (0, \infty) \setminus \mathcal{N} \), the inverse subordinator \( E_t \) has a density function given by

\[
\frac{d}{dr} \mathbb{P}(E_t \leq r) = \int_0^t w(t-s) \bar{p}(r,s) \, ds, \quad r > 0.
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We do not know a prior if $u(T - t, x)$ is in the domain of the generator $-\partial_t^w + \mathcal{L}$.

To rigorously prove these formulas, we use a different approach by studying the properties of subordinator and inverse subordinator. Here is an example.

**Theorem (C. 2017)**

There is a Borel set $\mathcal{N} \subset (0, \infty)$ having zero Lebesgue measure such that for every $t \in (0, \infty) \setminus \mathcal{N}$, the inverse subordinator $E_t$ has a density function given by

$$\frac{d}{dr} \mathbb{P}(E_t \leq r) = \int_0^t w(t - s) \overline{p}(r, s) ds, \quad r > 0.$$
For $a > 0$, let $f = 1_{[0,a]}$.

\[
\frac{d}{dt} \mathbb{P}(S_t \leq a) = \frac{d}{dt} P_t^{(S)} f(0) = \mathcal{L}^{(S)} P_t^{(S)} f(0)
\]
\[
= \int_0^\infty \left( P_t^{(S)} f(z) - P_t^{(S)} f(0) \right) \nu(dz)
\]
\[
= \int_0^\infty \mathbb{P}(a - z < S_t \leq a) dw(z)
\]
\[
= - \int_0^\infty w(z) dz \mathbb{P}(a - z < S_t \leq a)
\]
\[
= - \int_0^a w(z) \overline{p}(t, a - z) dz.
\]

(Used integration by parts formula and the fact that \(\lim_{z \downarrow 0} zw(z) = 0\) and \(\lim_{z \to \infty} w(z) = 0\).) Then \(\mathbb{P}(E_s \leq r) = \mathbb{P}(S_r > s)\) gives the formula.

Zhen-Qing Chen  University of Washington  Foundational Solutions to Time fractional Poisson equations
The indicator function \( f = 1_{[0,a]} \) is not in the domain of the Feller generator of the subordinator. Thus needs a different proof. Take Laplace transform on both sides to verify.
(i) When \( \{S_t; t \geq 0\} \) is a \( \beta \)-subordinator with \( 0 < \beta < 1 \) with Laplace exponent \( \phi(\lambda) = \lambda^\beta \), Then \( S_t \) has no drift (i.e. \( \kappa = 0 \)) and its Lévy measure is \( \mu(dx) = \frac{\beta}{\Gamma(1-\beta)} x^{-(1+\beta)} dx \). Hence

\[
w(x) := \mu(x, \infty) = \int_x^\infty \frac{\beta}{\Gamma(1-\beta)} y^{-(1+\beta)} dy = \frac{x^{-\beta}}{\Gamma(1-\beta)}.
\]

Thus the time fractional derivative \( \partial_t^w f \) is exactly the Caputo derivative of order \( \beta \). In this case, our Theorem recovers the main result of Baeumer-Meerschaert (2001) and Meerschaert-Scheffler (2004).
(ii) A truncated $\beta$-stable subordinator \( \{S_t; t \geq 0\} \) is driftless and has Lévy measure

\[
\mu_\delta(dx) = \frac{\beta}{\Gamma(1 - \beta)} x^{-(1+\beta)} 1_{(0,\delta]}(x)dx
\]

for some $\delta > 0$. In this case,

\[
w_\delta(x) := \mu_\delta(x, \infty) = 1_{\{0 < x \leq \delta\}} \int_x^\delta \frac{\beta}{\Gamma(1 - \beta)} y^{-(1+\beta)} dy
\]

\[
= \frac{1}{\Gamma(1 - \beta)} \left( x^{-\beta} - \delta^{-\beta} \right) 1_{(0,\delta]}(x).
\]

The corresponding the fractional derivative is

\[
\partial_t^{w_\delta} f(t) := \frac{1}{\Gamma(1 - \beta)} \frac{d}{dt} \int_{(t-\delta)^+}^{t} \left( (t - s)^{-\beta} - \delta^{-\beta} \right) (f(s) - f(0)) ds.
\]
Clearly, as \( \lim_{\delta \to \infty} w_\delta(x) = w(x) := \frac{1}{\Gamma(1-\beta)} x^{-\beta} \). Consequently, \( \partial_t^{\omega \delta} f(t) \to \partial_t^w f(t) \), the Caputo derivative of \( f \) of order \( \beta \), in the distributional sense as \( \delta \to 0 \). Using the probabilistic representation in the main Theorem, one can deduce that as \( \delta \to \infty \), the solution to the equation \( \partial_t^{\omega \delta} u = \mathcal{L} u \) with \( u(0, x) = f(x) \) converges to the solution of \( \partial_t^\beta u = \mathcal{L} u \) with \( u(0, x) = f(x) \).
(iii) If we define

\[ \eta_{\delta}(r) = \frac{\Gamma(2 - \beta)}{\beta} \delta^{\beta - 1} w_{\delta}(r) = (1 - \beta) \delta^{\beta - 1} \left( x^{-\beta} - \delta^{-\beta} \right) 1_{(0,\delta]}(x), \]

then \( \eta_{\delta}(r) \) converges weakly to the Dirac measure concentrated at 0 as \( \delta \to 0 \). So \( \partial_t^{\eta_{\delta}} f(t) \) converges to \( f'(t) \) for every differentiable \( f \). It can be shown that the subordinator corresponding to \( \eta_{\delta} \), that is, subordinator with Lévy measure

\[ \nu_{\delta}(dx) := \frac{(1 - \beta) \delta^{\beta - 1}}{\beta} x^{-(1+\beta)} 1_{(0,\delta]}(x) dx, \]

converges as \( \delta \to 0 \) to deterministic motion \( t \) moving at constant speed 1. Using the main Theorem, one can show that the solution to the equation \( \partial_t^{\eta_{\delta}} u(t, x) = \mathcal{L} u(t, x) \) with \( u(0, x) = f(x) \) converges to the solution of the heat equation \( \partial_t u = \mathcal{L} u \) with \( u(0, x) = f(x) \).


Thank you!