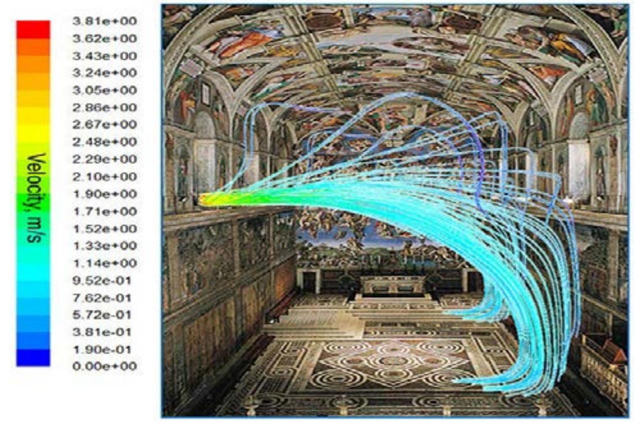
High Order Approximations for Control & Estimation of PDE Systems

John A. Burns and James Cheung

Interdisciplinary Center for Applied Mathematics

Virginia Tech

Blacksburg, Virginia 24061-0531

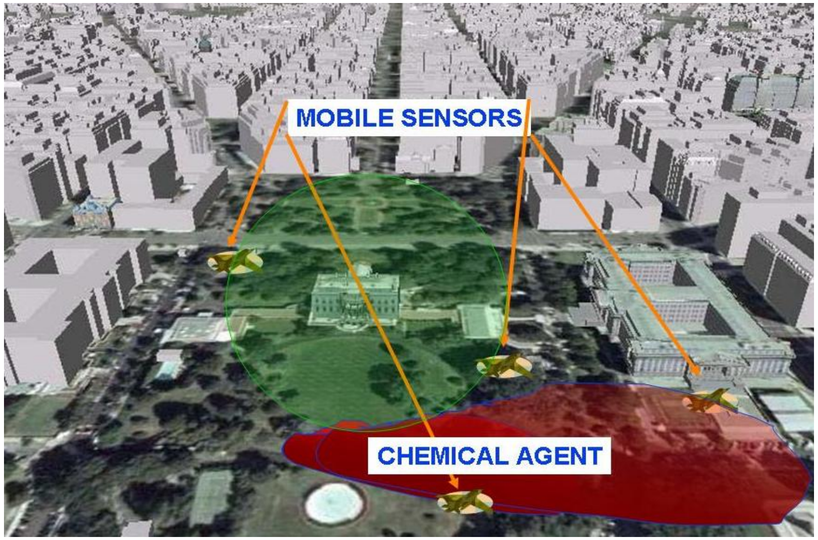


WORKSHOP ON DYNAMICS, CONTROL AND NUMERICS FOR FRACTIONAL PDEs

December 5 – 7, 2018

San Juan, Puerto Rico

Motivation: From Energy Independence to Homeland Security to National Defense

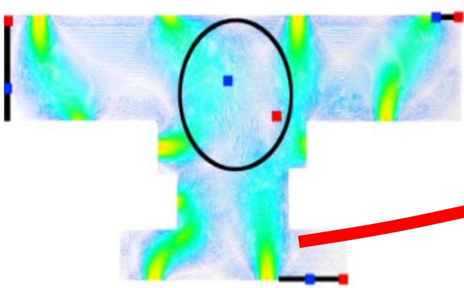


Mathematical Characteristics Common to All

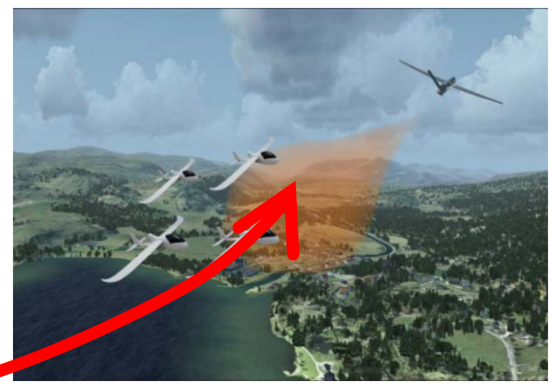
Heat Exchanger
Thermal + Fluid

Wing Flutter
Structure + Fluid

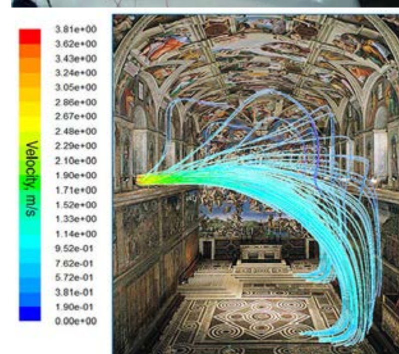
Air Conditioning
Thermal + Fluid
+ Mechanical + …



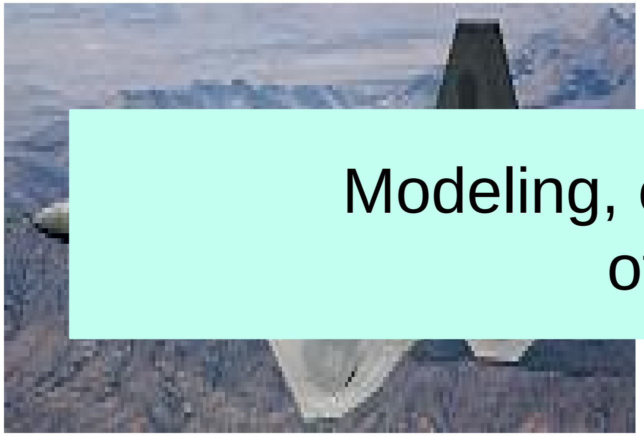
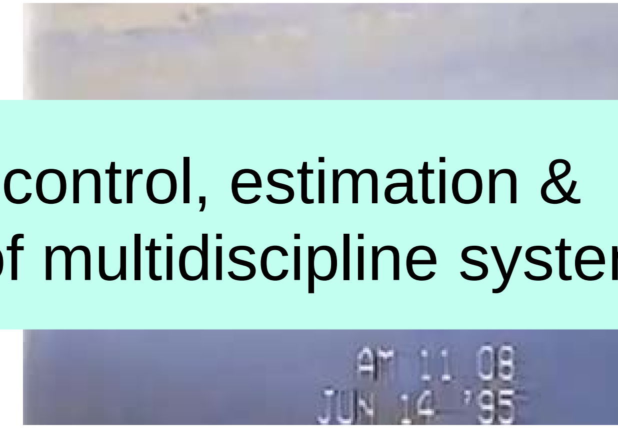
Optimal Sensor Location
Thermal + Fluid

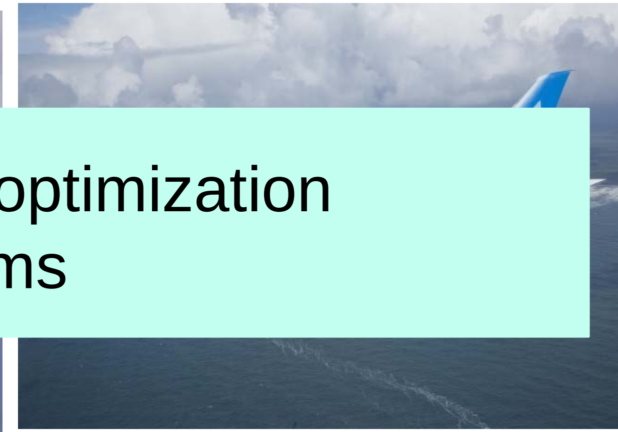


Plume Estimation
Convection-Diffusion
+ Flight Dynamics



Mathematical Characteristics Common to All

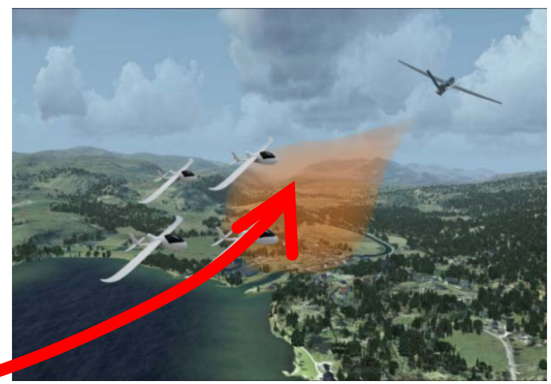


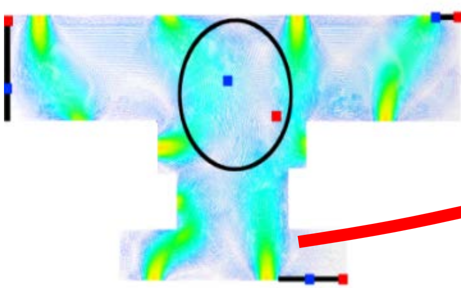
Modeling, control, estimation & optimization
of multidiscipline systems

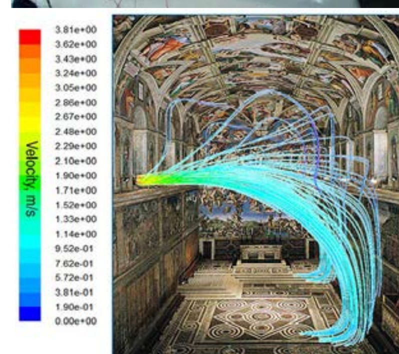
Heat Exchanger
Thermal + Fluid

Wing Flutter
Structure + Fluid

Air Conditioning
Thermal + Fluid
+ Mechanical + …





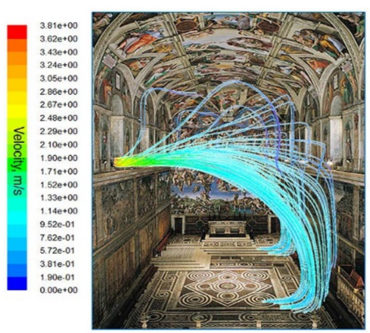


Optimal Sensor Location
Thermal + Fluid

Plume Estimation
Convection-Diffusion
+ Flight Dynamics

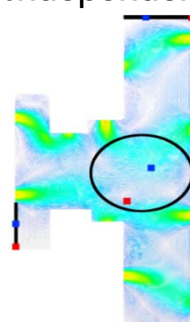
People & Problems

- Virginia Tech: Jeff Borggaard, Gene Cliff, Terry Herdman, Veronica Gheorghe, Zach Grigorian, James Cheung
- Carnegie Mellon: Larry Biegler
- Humboldt, Berlin: Carlos N. Rautenberg
- Oklahoma State: Weiwei Hu
- UT CCS: Trevor Bailey, Qingfan Zeng, Degang Fu, Rui Huang, Clas Jacobson
- WPI: Michael Demetriou, Nikolaos Gatsonis



PDE Optimization & Control of Dynamic Sensor Systems ...

Energy Independence



Homeland Security

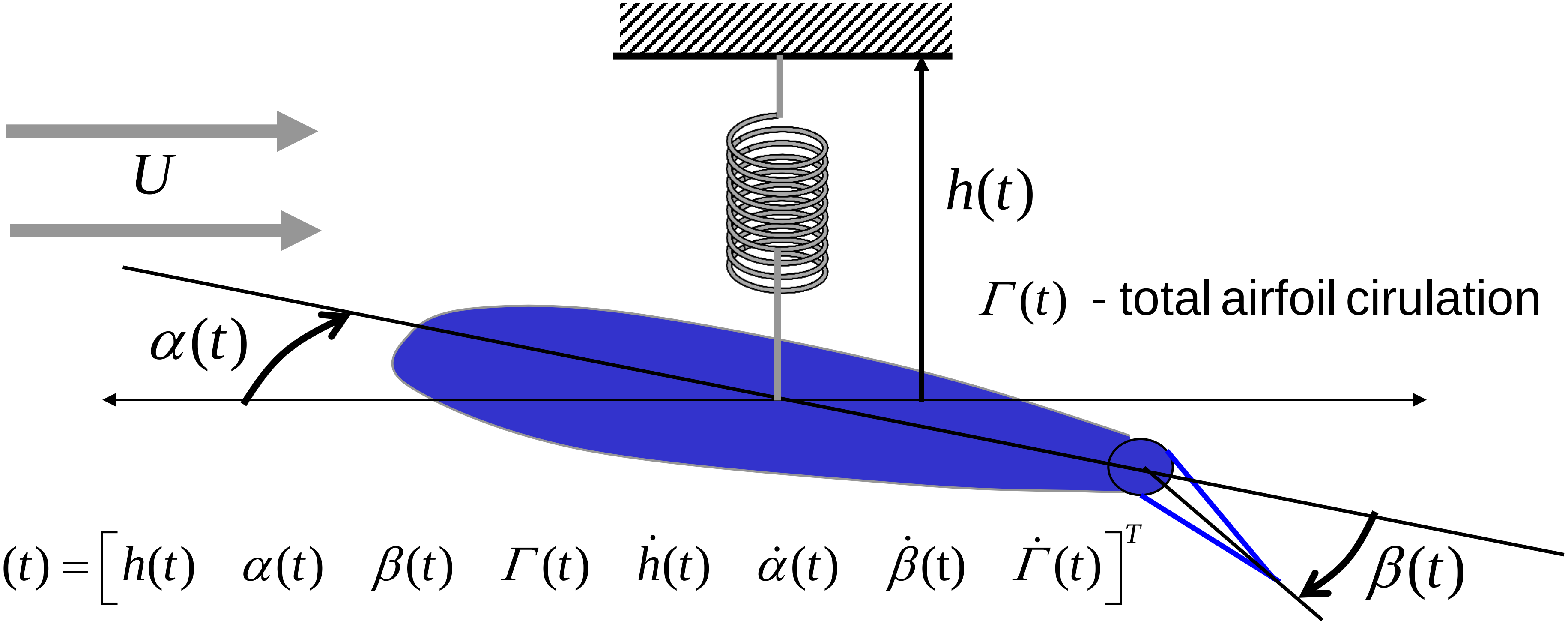
Defense

computational complexity

Combine physics based models, higher order methods and optimization for real time estimation and control

Wing Flutter: Aeroelasticity

Simplest Linear Flutter Model



$$\dot{x}(t) = \begin{bmatrix} h(t) & \alpha(t) & \beta(t) & \Gamma(t) & \dot{h}(t) & \dot{\alpha}(t) & \dot{\beta}(t) & \dot{\Gamma}(t) \end{bmatrix}^T$$

$$\frac{d}{dt}\left[\mathbf{A}x(t) + \int_{-\infty}^{0} \mathbf{D}(s)x(t+s)ds\right] = \left[\mathbf{L}x(t) + \int_{-\infty}^{0} \mathbf{K}(s)x(t+s)ds\right] + \mathbf{G}u(t)$$

$$\mathbf{A}_{8,8} = 0 \qquad \mathbf{D}_{8,8}(s) = \sqrt{\frac{Us-2}{s}}$$ weakly singular kernel

non-atomic neutral <u>delay differential equation</u>

Simple Aeroelastic Models

A. V. Balakrishnan, "*Active Control of Airfoils in Unsteady Aerodynamics*", Applied Mathematics and Optimization", 4 (1978), 171-195.

J. A. Burns, E. M. Cliff and T. L. Herdman, "*A State Space Model for an Aeroelastic System*", 22nd IEEE Confeence on Decision and Control, (1983), 1074-1077.

J. A. Burns, T. L. Herdman and H. Stech, "*Linear functional differential equations as semigroups on product spaces*", SIAM J. Math. Anal. 14 (1983), 98-116.

J. A. Burns, E. M. Cliff and R. K. Powers , "*Modeling and Control of an Aeroelastic System*", Fourth IFAC Symposium on Control of Distributed Parameter Systems, UCLA, July 1986, 6-11.

H. Ozbay and J. Turi,, "*Robust Control of an Aeroelastic System Modeled by a Singular Integro-Differential Equation*", Fourth IFAC Symposium on Control of Distributed Parameter Systems, UCLA, July 1986, 2682-2686.

J. A. Burns and T. L. Herdman, "*Neutral Functional Integro- Differential Equations with Weakly Singular Kernels*", JMAA, 1990, 1074-1077.

D. Swinney, "*A Fractional Calculus Model of Aeroelasticity*", PhD Thesis, AFIT, Dayton, OH, 1989.

R. Bagley, D. Swinney and K. Griffin , "*Fractional Order Calculus Model of the Generalized Theodorsen Functions*", J. Aircraft, 30 (1993), 1074-1077.

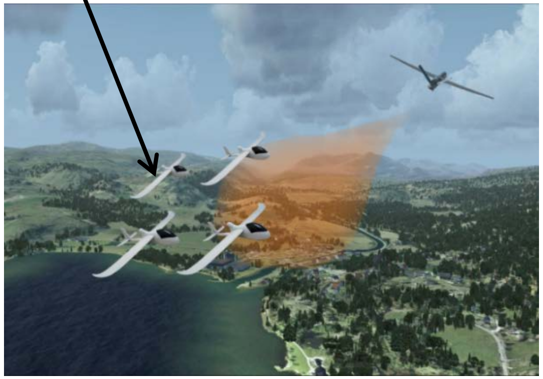
BIG advances in past 30 years - I. Lasiecka...

Optimization of Real Time Dynamic Sensor Systems

$$z_t(t,x) = \nabla \cdot \left[K(t,x)\nabla z(t,x)\right] - \nabla \cdot \left[\mathbf{V}(t,x)z(t,x)\right] + g(x)\eta(t), \;\; x \in \Omega, \;\; t > 0$$

$$y_j(t) = \iiint_{B(\vec{w}_j(t),\delta)\cap\Omega} h(t,x)z(t,x)dx + D\xi(t)$$

$$\vec{w}_j(t) = (x_j(t), y_j(t), z_j(t))$$



$$\frac{d^2\vec{w}_j(t)}{dt^2} = f_j((x_j(t), y_j(t), z_j(t), u_j(t)) = f_j(\vec{w}_j(t), u_j(t))$$

$$\dot{z}(t) = \mathcal{A}(\mathbf{V})z(t) + \mathcal{G}\eta(t) \qquad \boldsymbol{y}(t) = \mathcal{C}(\overrightarrow{\mathbf{w}}(\cdot))z(t) + \mathcal{D}\xi(t)$$

State estimator (observer) …

$$\dot{z}_e(t) = [\mathcal{A}(\mathbf{V}) - \mathcal{F}(t)\mathcal{C}(t)]z_e(t) + \mathcal{F}(t)\boldsymbol{y}(t) + \mathcal{G}\eta(t)$$

$$= [\mathcal{A}(\mathbf{V}) - \mathcal{F}(t,\overrightarrow{\mathbf{w}}(t))\mathcal{C}(t,\overrightarrow{\mathbf{w}}(t))]z_e(t) + \mathcal{F}(t,\overrightarrow{\mathbf{w}}(t))\boldsymbol{y}(t) + \mathcal{G}\eta(t)$$

$$\frac{d^2\overrightarrow{\mathbf{w}}(t)}{dt^2} = \boldsymbol{f}(\overrightarrow{\mathbf{w}}(t), \vec{\boldsymbol{u}}(t))$$

control / estimation and optimization of a composite system

Problem: Find the observer operator $\mathcal{F}^{opt}(t) = \mathcal{F}^{opt}(t,\overrightarrow{\mathbf{w}}(t),\vec{\boldsymbol{u}}(t))$ to minimize

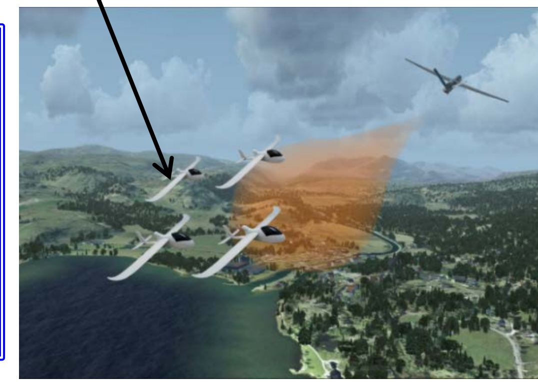
$$E = \max_{0\le t\le T_F} \left\|z(t) - z_e(t)\right\| \quad \text{or} \quad E = \int_0^{T_F} \left\|z(s) - z_e(s)\right\|^2 ds \quad \text{or} \quad E = \mathbb{E}\left(\int_0^{T_F} \left\|z(s) - z_e(s)\right\|^2 ds\right)$$

Optimization of Real Time Dynamic Sensor Systems

$$z_t(t,x) = \nabla\cdot\left[K(t,x)\nabla z(t,x)\right] - \nabla\cdot\left[\mathbf{V}(t,x)z(t,x)\right] + g(x)\eta(t), \quad x\in\Omega,\ t>0$$

$$y_j(t) = \iiint_{B(\vec{w}_j(t),\delta)\cap\Omega} h(t,x)z(t,x)\,dx + D\xi(t)$$

$$\vec{w}_j(t) = (x_j(t), y_j(t), z_j(t))$$



$$\dot{z}_e(t) = [\mathcal{A}(\mathbf{V}) - \mathcal{F}(t)\mathcal{C}(t)]z_e(t) + \mathcal{F}(t)\boldsymbol{y}(t) + \mathcal{G}\eta(t)$$

$$= [\mathcal{A}(\mathbf{V}) - \mathcal{F}(t,\mathbf{w}(t))\mathcal{C}(t,\mathbf{w}(t))]z_e(t) + \mathcal{F}(t,\mathbf{w}(t))\boldsymbol{y}(t) + \mathcal{G}\eta(t)$$

$$\frac{d^2\mathbf{w}(t)}{dt^2} = \boldsymbol{f}(\mathbf{w}(t), \boldsymbol{u}(t))$$

REAL TIME estimation, control & optimization ?

IMPOSSIBLE ! … MAYBE NOT ?

Egorova, T., Gatsonis, N.A., and Demetriou, M.A., "*Estimation of Gaseous Plume Concentration with an Unmanned Aerial Vehicle*", Vol. 39, No. 6, pp. 1314-1324, Journal of Guidance, Control, and Dynamics, 2016.

Demetriou, M. A., Gatsonis, N.A., and Court, J., "*A coupled controls-computational fluids approach for the estimation of the concentration from a moving gaseous source in a 2D domain with a Lyapunov-guided sensing aerial vehicle*", IEEE Transactions on Control Systems Technology, Vol. 22, 3, pp. 853 – 867, 2014

Moving Intruder & “Sub-Optimal” Luenberger Estimator

Intruder with plume (Left)

Estimated Plume by a UAV carrying a noiseless sensor (Right)

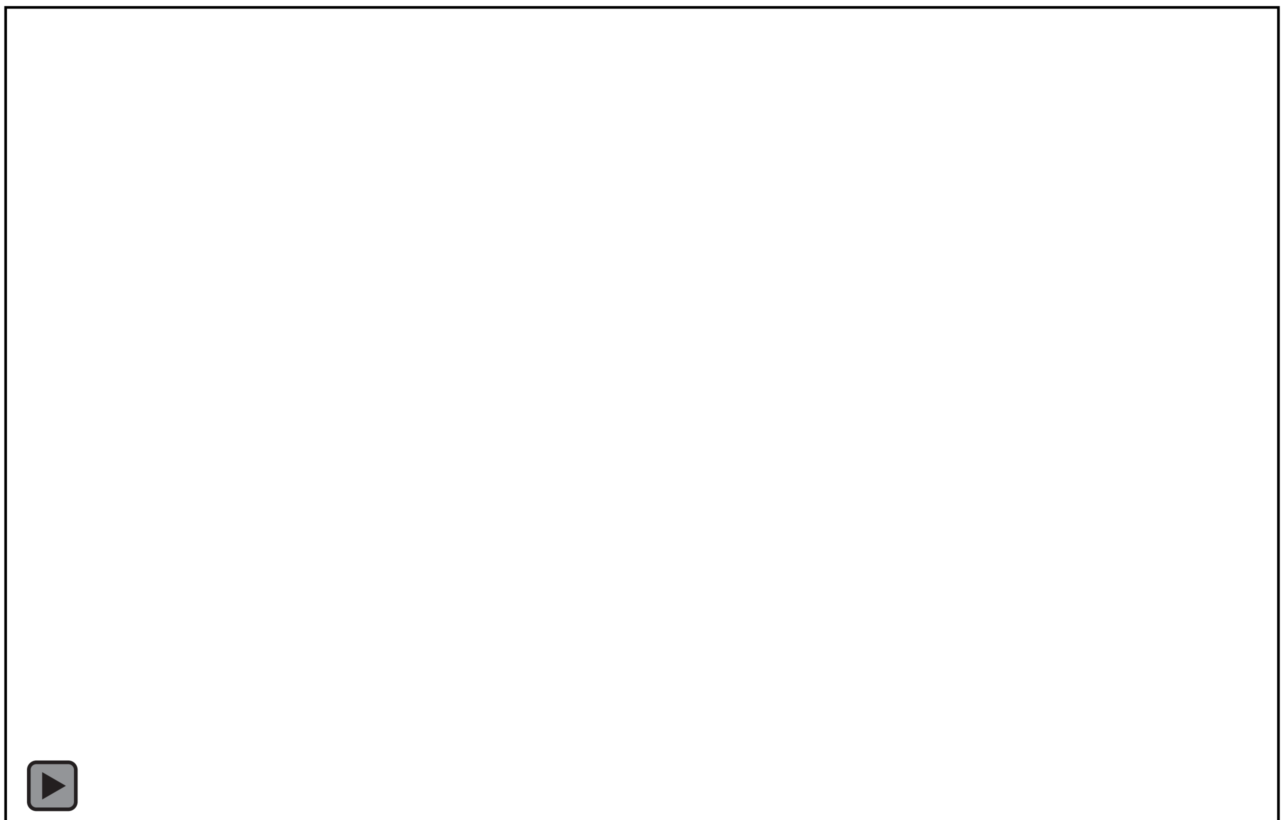
Demetriou & Gatsonis (WPI)

Moving Sensor & Sub-Optimal Luenberger Estimator with Grid Adaptation

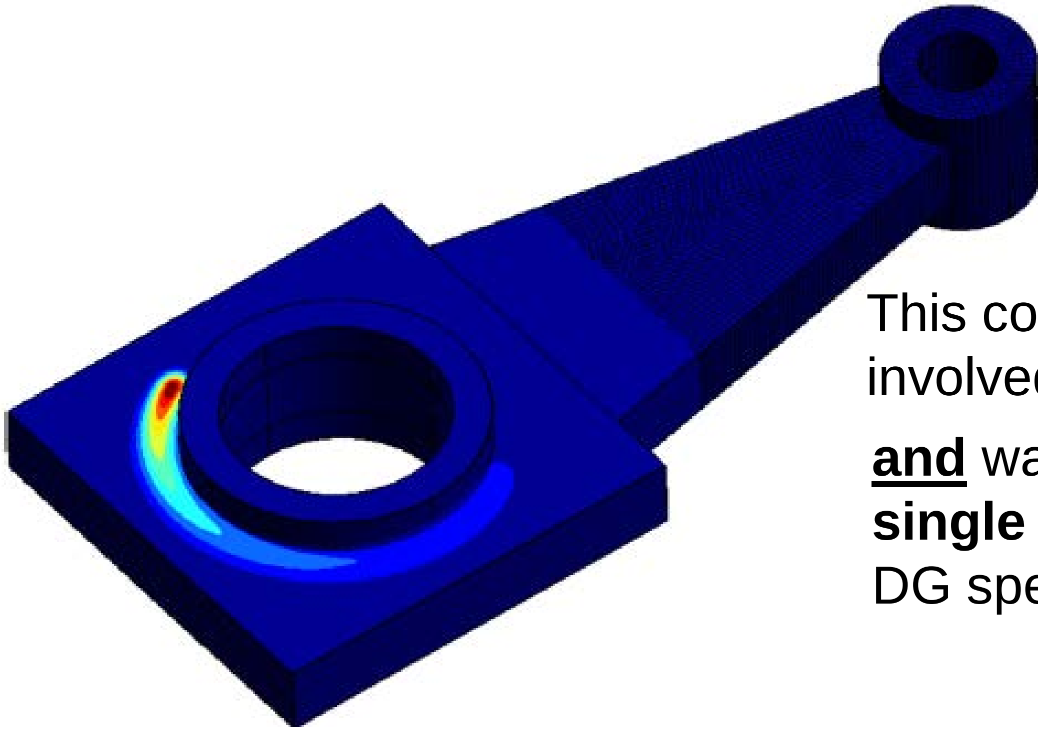
BUT, REAL TIME OPTIMAL ESTIMATION IS IMPOSSIBLE - RIGHT?

MAYBE NOT - WITH HIGHER ORDER (DG – *hp*) *METHODS*
AND
GPU COMPUTING ! **Demetriou & Gatsonis (WPI)**

Adaptive multi-grid / domain decomposition method



Computing (Parallel, DG & Spectral Element on GPUs)



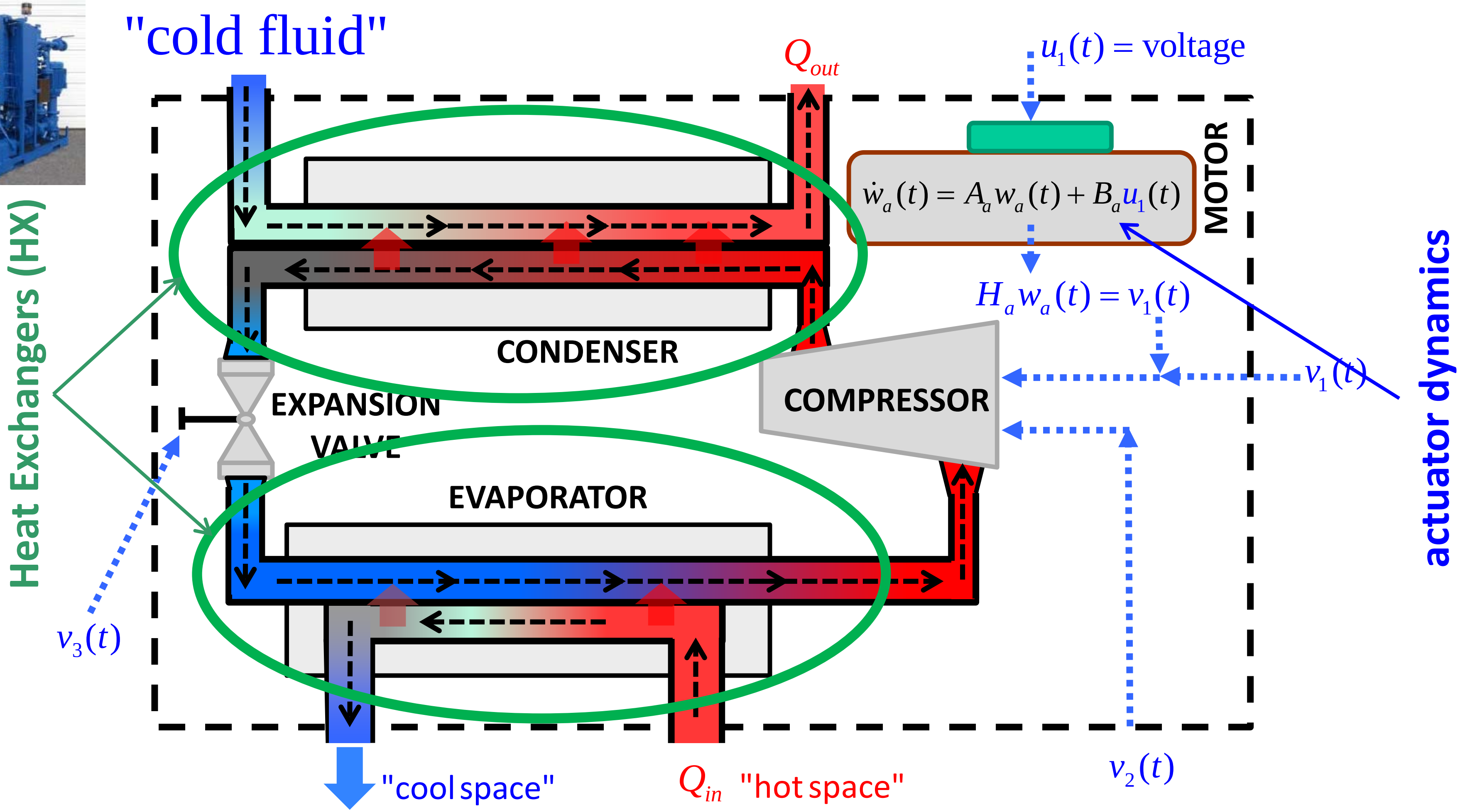
This computation (a 3D heat equation) involved more than 20 million equations

and was solved in **a few seconds on a single $500 GPU** processor by using a DG spectral element method !

J.-F. Remacle, R. Gandham and T. Warburton, "GPU accelerated spectral finite elements on all-hex meshes", Journal of Computational Physics ***324*** *(2016), 246–257.*

M. Kohler and J. Saak, "On GPU acceleration of common solvers for quasi-triangular generalized Lyapunov equations", J. Parallel Computing, ***57*** *(2016), 212–221.*

Heat Exchanger: Basic Vapor Compression System



$v_1(t) = P(t)$ - compressor input power (kw)

$v_2(t) = V_{des}(t)$ - compressor volume flow rate (m^2/s)

$v_3(t) = C_f(t)$ - venturi area valve (m^2)

DYNAMIC MODELING A CHALLENGE

WELL-POSEDNESS OF CLOSED CYCLE IS NOT OBVIOUS

$ODE_1 \Rightarrow PDE_1 \Rightarrow ODE_2 \Rightarrow PDE_2 \Rightarrow ODE_1$: Closed Cycle

$$\theta(t,s) = v_1(t+s)$$

$$\theta(t,0) = v_1(t)$$

$$\frac{\partial}{\partial t}\theta(t,s) = \frac{\partial}{\partial s}\theta(t,s)$$

$$u_2(t) = v_2(t)$$

$$v_1(t) = y_1(t)$$

$$v_2(t) = \theta(t,-L) = c_1 w(t-L)$$

$s=-L$ $\qquad s \qquad$ $s=0$

$$\eta(t) \in \mathbb{R}^1$$

$$\dot{\eta}(t) = a_2\eta(t) + b_2 u_2(t)$$

$$y_2(t) = c_2\eta(t)$$

$$\dot{w}(t) = a_1 w(t) + b_1 u_1(t)$$

$$y_1(t) = c_1 w(t)$$

$$w(t) \in \mathbb{R}^1$$

$$\omega(t,s) = y_2(t-(L+s))$$

$$u_1(t) = y_2(t-L) = c_2\eta(t-L)$$

$$\omega(t,-L) = y_2(t) = c_2\eta(t)$$

$$\frac{\partial}{\partial t}\omega(t,s) = -\frac{\partial}{\partial s}\omega(t,s)$$

$$\omega(t,0) = y_2(t-L)$$

$s=-L$ $\qquad s \qquad$ $s=0$

ICAM ODE$_1$ ⇒ PDE$_1$ ⇒ ODE$_2$ ⇒ PDE$_2$ ⇒ ODE$_1$: Closed Cycle

$$\dot{w}(t) = a_1 w(t) + b_1 c_2 \eta(t-L)$$

$$\dot{\eta}(t) = a_2 \eta(t) + b_2 c_1 w(t-L)$$

$$\boldsymbol{x}(t) = \begin{bmatrix} w(t) \\ \eta(t) \end{bmatrix}$$

$$\dot{x}(t) = \begin{bmatrix} \dot{w}(t) \\ \dot{\eta}(t) \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} w(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} 0 & b_1 c_2 \\ b_2 c_1 & 0 \end{bmatrix} \begin{bmatrix} w(t-L) \\ \eta(t-L) \end{bmatrix}$$

AGAIN, a <u>delay</u> differential equation $\quad \dot{\boldsymbol{x}}(t) = \mathbf{F}_1 \boldsymbol{x}(t) + \mathbf{F}_2 \boldsymbol{x}(t-L)$

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{\varphi}(\cdot) \\ \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} \varphi_1(\cdot) \\ \varphi_2(\cdot) \\ w \\ \eta \end{bmatrix} \in Z \triangleq L_2(-L,0) \times L_2(-L,0) \times \left(\mathbb{R}^1 \times \mathbb{R}^1\right)$$

$$D(\mathcal{A}) = \left\{ \begin{bmatrix} \boldsymbol{\varphi}(\cdot) \\ \boldsymbol{x} \end{bmatrix} : \boldsymbol{\varphi}(\cdot) \in H^1(-L,0)^2, \boldsymbol{\varphi}(0) = \boldsymbol{x} \right\} \qquad \mathcal{A}\left(\begin{bmatrix} \boldsymbol{\varphi}(\cdot) \\ \boldsymbol{x} \end{bmatrix} \right) = \begin{bmatrix} \boldsymbol{\varphi}'(\cdot) \\ \mathbf{F}_1 \eta + \mathbf{F}_2 \boldsymbol{\varphi}(-L) \end{bmatrix}$$

$ODE_1 \Rightarrow PDE_1 \Rightarrow ODE_2 \Rightarrow PDE_2 \Rightarrow ODE_1$: Closed Cycle

$$\dot{x}(t) = \begin{bmatrix} \dot{w}(t) \\ \dot{\eta}(t) \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} w(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} 0 & b_1 c_2 \\ b_2 c_1 & 0 \end{bmatrix} \begin{bmatrix} w(t-L) \\ \eta(t-L) \end{bmatrix}$$

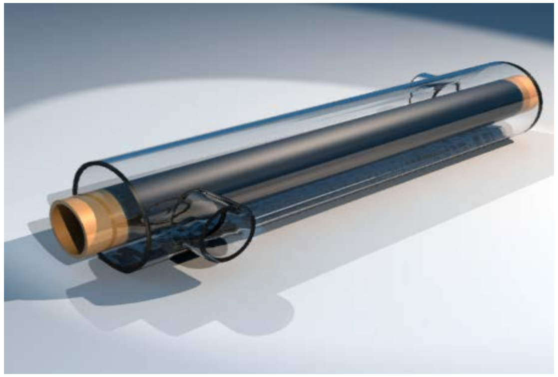
$$\theta(t,s) = c_1 w(t+s) \qquad \omega(t,s) = c_2 \eta(t-(L+s))$$

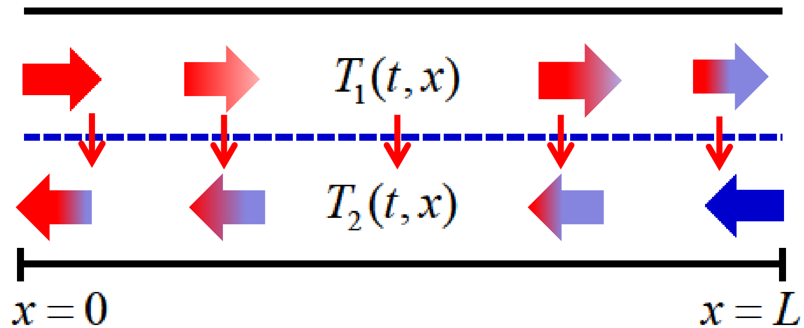
- This system is closely related to a delay differential equation
- Numerical methods developed for delay systems offer insight
- Much more difficult if the ODE systems are vector systems
- Steady state behavior of closed cycle cannot be implied by linking steady state solutions of each component

Example shows that these type of systems are rarely self-adjoint and in fact can be highly non-normal.

Interest in high order "dual convergent" approximation schemes for optimal estimation and control.

A Simple (Linear) Counter Flow Heat Exchanger





$$\frac{\partial T_1(t,x)}{\partial t} = \mu_1 \frac{\partial^2 T_1(t,x)}{\partial x^2} - v_1 \frac{\partial T_1(t,x)}{\partial x} + h_1\left(T_2(t,x) - T_1(t,x)\right), \quad T_1(t,0) = \boxed{g(t)} \quad \mu_1 \frac{\partial T_1(t,L)}{\partial x} = 0$$

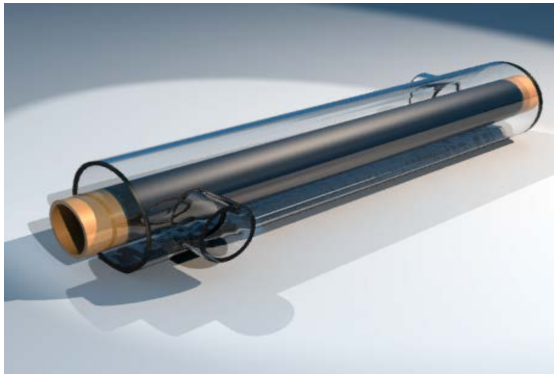
$$\frac{\partial T_2(t,x)}{\partial t} = \mu_2 \frac{\partial^2 T_2(t,x)}{\partial x^2} + v_2 \frac{\partial T_2(t,x)}{\partial x} + h_2\left(T_1(t,x) - T_2(t,x)\right), \quad T_2(t,L) = 0, \quad -\mu_2 \frac{\partial T_2(t,0)}{\partial x} = 0$$

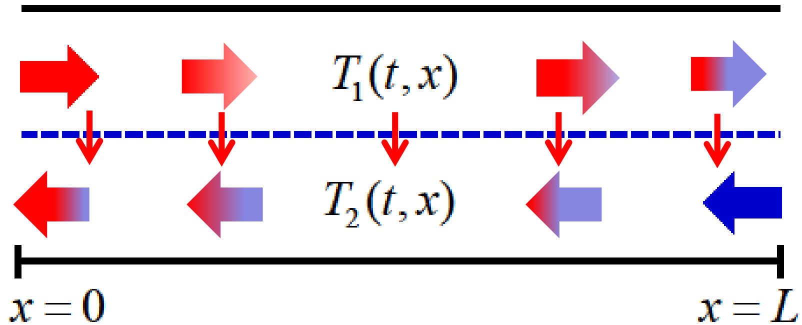
possible controls - disturbances

actuator dynamics

$$u(t) \longrightarrow \boxed{\begin{aligned} \dot{w}(t) &= \mathbf{A}_a w(t) + \mathbf{B}_a u(t) \\ g(t) &= \mathbf{H}_a w(t) \end{aligned}} \longrightarrow g(t)$$

A Simple (Linear) Counter Flow Heat Exchanger





$$\frac{\partial T_1(t,x)}{\partial t} = \mu_1 \frac{\partial^2 T_1(t,x)}{\partial x^2} - v_1 \frac{\partial T_1(t,x)}{\partial x} + h_1\left(T_2(t,x) - T_1(t,x)\right), \quad T_1(t,0) = \boxed{g(t)} \quad \mu_1 \frac{\partial T_1(t,L)}{\partial x} = 0$$

$$\frac{\partial T_2(t,x)}{\partial t} = \mu_2 \frac{\partial^2 T_2(t,x)}{\partial x^2} + v_2 \frac{\partial T_2(t,x)}{\partial x} + h_2\left(T_1(t,x) - T_2(t,x)\right), \quad T_2(t,L) = 0, \quad -\mu_2 \frac{\partial T_2(t,0)}{\partial x} = 0$$

possible controls - disturbances

actuator dynamics

$$u(t) \longrightarrow \boxed{\begin{aligned} \dot{w}(t) &= \mathbf{A}_a w(t) + \mathbf{B}_a u(t) \\ g(t) &= \mathbf{H}_a w(t-r) \end{aligned}} \longrightarrow g(t)$$

A Simple (Linear) Counter Flow Heat Exchanger

CHANGE OF VARIABLES $\theta_1(t,x) = T_1(t,x) - g(t) \qquad \theta_2(t,x) = T_2(t,x)$

$$\cancel{\frac{\partial \theta_1(t,x)}{\partial t}} = \mu_1 \cancel{\frac{\partial^2 \theta_1(t,x)}{\partial x^2}} - \boldsymbol{v_1} \frac{\partial \theta_1(t,x)}{\partial x} + h_1\left(\theta_2(t,x) - \theta_1(t,x)\right) + F_1 w(t) + B_1 \mathbf{u}(t) \qquad \theta_1(t,0) = \mu_1 \cancel{\frac{\partial \theta_1(t,L)}{\partial x}} = 0$$

$$\cancel{\frac{\partial \theta_2(t,x)}{\partial t}} = \mu_2 \cancel{\frac{\partial^2 \theta_2(t,x)}{\partial x^2}} + \boldsymbol{v_2} \frac{\partial \theta_2(t,x)}{\partial x} + h_2\left(\theta_1(t,x) - \theta_2(t,x)\right) + F_2 w(\mathrm{t}) \qquad -\mu_2 \cancel{\frac{\partial \theta_2(t,0)}{\partial x}} = \theta_2(t,L) = 0$$

$$\dot{w}(t) = \mathbf{A}_a w(t) + \mathbf{B}_a \mathbf{u}(t)$$

$$F_1 = -[\mu_1 \mathbf{H}_a + \mathbf{H}_a \mathbf{A}_a] \qquad F_2 = +\mu_2 \mathbf{H}_a \qquad B_1 = -\mathbf{H}_a \mathbf{B}_a$$

STATE OF THE ART IN INDUSTRY: FINITE VOLUME METHOD ON
"SIMPLIFIED STEADY STATE MODELS"

CHALLENGES: NUMERICAL SCHEMES FOR CONVECTION-DIFFUSION SYSTEMS (& FULL NONLINEAR CONSERVATION LAWS) THAT ARE:

(a) VALID ACROSS RANGE OF OPERATIONS (e.g., LOW FLOW)
(b) SUITABLE FOR OPTIMIZATION & CONTROL DESIGN
(c) <u>HIGH ORDER & SUITABLE FOR GPU COMPUTING</u>

Combined FE-FV Approximations

$$\frac{\partial\theta_1(t,x)}{\partial t} = \mu_1 \frac{\partial^2\theta_1(t,x)}{\partial x^2} - v_1 \frac{\partial\theta_1(t,x)}{\partial x} + h_1\left(\theta_2(t,x)-\theta_1(t,x)\right) + F_1 w(t) + B_1 u(t) \qquad \theta_1(t,0) = \mu_1 \frac{\partial\theta_1(t,L)}{\partial x} = 0$$

$$\frac{\partial\theta_2(t,x)}{\partial t} = \mu_2 \frac{\partial^2\theta_2(t,x)}{\partial x^2} + v_2 \frac{\partial\theta_2(t,x)}{\partial x} + h_2\left(\theta_1(t,x)-\theta_2(t,x)\right) + F_2 w(\mathrm{t}) \qquad -\mu_2 \frac{\partial\theta_2(t,0)}{\partial x} = \theta_2(t,L) = 0$$

$$\dot{w}(t) = \mathbf{A}_a w(t) + \mathbf{B}_a u(t)$$

The differential operators $P(\mu_j) = \mu_j \dfrac{\partial^2}{\partial x^2}$ are elliptic (and selfadjoint)

The differential operators $H(v_j) = v_j \dfrac{\partial}{\partial x}$ are NOT selfadjoint

Basic Idea : Approximate the selfadjoint operators $P(\mu_j)$ using finite elements and the non - selfadjoint operators $H(v_j)$ using a upwind finite volume scheme.

P. Deuring, R. Eymard, and M. Mildner, "*L^2-stability independent of diffusion for a finite element - finite volume discretization of a linear convection-diffusion equation*", SIAM Journal on Numerical Analysis, 53 (2015), 508--526

J. A. Burns and L. Zietsman, "*Control of a Thermal Fluid Heat Exchanger with Actuator Dynamics*", in Proceedings of the 55th IEEE Conference on Decision and Control, Las Vegas, NV, December, 2016, 3131–3136.

FVM IS "LOW ORDER" USE "HIGHER ORDER DG" or "*hp*" SCHEMES?

Combined FE-FV Approximations

Focus on the non-selfadjoint "hyperbolic" operators $H(v_j)$

$$\frac{\partial \theta_1(t,x)}{\partial t} = \mu_1 \frac{\partial^2 \theta_1(t,x)}{\partial x^2} - \boldsymbol{v_1} \frac{\partial \theta_1(t,x)}{\partial x} + h_1\big(\theta_2(t,x) - \theta_1(t,x)\big) + F_1 w(t) + B_1 u(t) \qquad \theta_1(t,0) = \mu_1 \frac{\partial \theta_1(t,L)}{\partial x} = 0$$

$$\frac{\partial \theta_2(t,x)}{\partial t} = \mu_2 \frac{\partial^2 \theta_2(t,x)}{\partial x^2} + \boldsymbol{v_2} \frac{\partial \theta_2(t,x)}{\partial x} + h_2\big(\theta_1(t,x) - \theta_2(t,x)\big) + F_2 w(\mathrm{t}) \qquad -\mu_2 \frac{\partial \theta_2(t,0)}{\partial x} = \theta_2(t,L) = 0$$

$$\dot{w}(t) = \mathbf{A}_a w(t) + \mathbf{B}_a u(t)$$

The differential operators $P(\mu_j) = \mu_j \dfrac{\partial^2}{\partial x^2}$ are elliptic (and selfadjoint)

The differential operators $H(v_j) = v_j \dfrac{\partial}{\partial x}$ are NOT selfadjoint

Use "higher order" (and dual convergent) DG & "*hp*" type schemes

- J. S. Hesthaven and T. Warburton. *Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications*. Springer Science & Business Media, 2007
- J. S. Hesthaven. *Numerical Methods for Conservation Laws:From Analysis to Algorithms*, SIAM, 2018.
- J. Cheung, *Overcoming Geometric Limitations in the Finite Element Method by Means of Polynomial Extension: Application to Second–Order Elliptic Boundary Value and Interface Problems*, PhD Thesis, Florida State University, 2018.

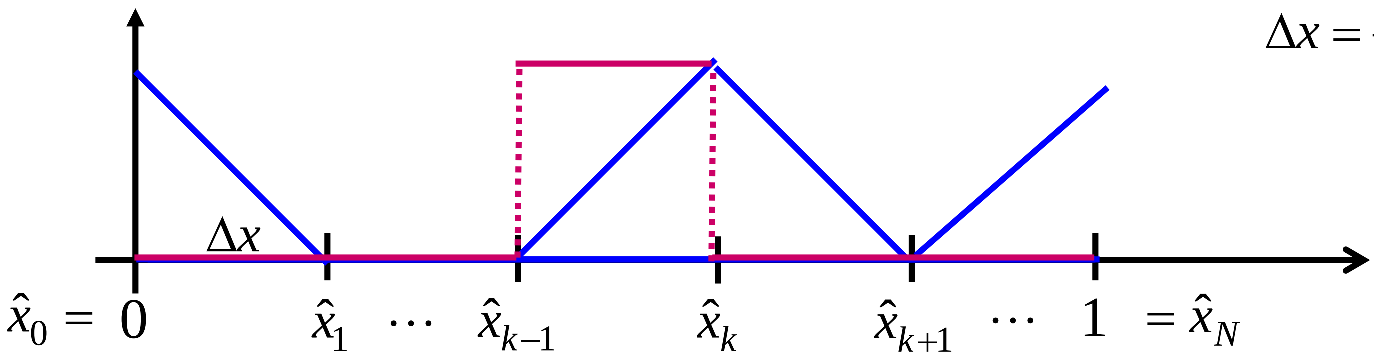
Related to an idea introduced by Ito and Kappel in 1991 (for delay equations)

- K. Ito and F. Kappel, "*A Uniformly Differentiable Approximation Scheme for Delay Systems Using Splines*", Applied Mathematics and Optimization, **23** (1991), 217--262 .

A “Higher Order” FE - C_0 Galerkin Scheme

$$\frac{\partial u(t,x)}{\partial t} = -\boldsymbol{v_1}\frac{\partial u(t,x)}{\partial x}\ ,\ 0<x<1\ ,\ t>0 \qquad u(t,0)=0 \qquad u(0,x)=\varphi(x)$$

$$\Delta x = \tfrac{1}{N}$$



$$\cancel{h_0(x)}$$

$$h_k(x)$$

$$h_N(x)$$

$$\chi_k(x) = \begin{cases} 1, & x \in (x_{k-1}, x_k] \\ 0, & \text{otherwise} \end{cases}$$

$$W^N = span\{h_k(\cdot) : k = 1,2,..,N\} \qquad Z^N = span\{\chi_k(\cdot) : k = 1,2,..,N\}$$

"Finite Volume - Upwind" Scheme

$$\frac{\partial u(t,x)}{\partial t} = -\boldsymbol{v_1}\frac{\partial u(t,x)}{\partial x}\text{ , } 0 < x < 1\text{ , } t > 0 \qquad u(t,0) = \eta \qquad u(0,x) = \varphi(x)$$

$$z^N(t) = \sum_{k=1}^{N} a_k(t)\chi_k(x)$$

$$(\Sigma_{FV}) \quad \dot{z}^N(t) = \mathcal{H}^N z^N(t) \in Z^N$$

Dual convergence holds, but low order /slow

For optimization; we want a "higher order"

and dual convergent scheme...

Take advantage of the Ito-Kappel idea ...

- K. Ito and F. Kappel, "*A Uniformly Differentiable Approximation Scheme for Delay Systems Using Splines*", Applied Mathematics and Optimization, **23** (1991), 217--262 .

Two Spline Schemes (FE)

$$\frac{\partial u(t,x)}{\partial t} = -\boldsymbol{v_1}\frac{\partial u(t,x)}{\partial x}\ ,\ 0<x<1\ ,\ t>0 \qquad u(t,0)=\eta \qquad u(0,x)=\varphi(x)$$

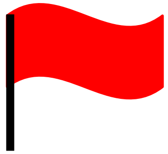
$$w^N(t)=\sum_{k=1}^{N} b_k(t)h_k(x) \in W^N$$

$$\mathcal{P}_{FE}^N : Z \to W^N$$

"standard" FE scheme

$$(\Sigma_{FE}) \qquad \dot{w}^N(t) = \mathcal{P}_{FE}^N \mathcal{H} w^N(t) \in W^N$$

... dual convergence fails



Two Spline Schemes (IK)

$$\frac{\partial u(t,x)}{\partial t} = -\boldsymbol{v_1} \frac{\partial u(t,x)}{\partial x} \text{ , } 0 < x < 1 \text{ , } t > 0 \qquad u(t,0) = \eta \qquad u(0,x) = \varphi(x)$$

$$w^N(t) = \sum_{k=1}^{N} b_k(t) h_k(x) \in W^N$$

$$\dot{w}^N(t) = \sum_{k=1}^{N} \dot{b}_k(t) h_k(x) \in W^N \qquad w_x^N(t) = \sum_{k=1}^{N} b_k(t) h_k'(x) \in Z^N$$

$$\mathcal{P}_{FV}^N : Z \to Z^N$$

$$(\Sigma_{CG}) \qquad \mathcal{P}_{FV}^N \dot{w}^N(t) = \mathcal{H} w^N(t) \quad \in Z^N$$

Higher order AND dual convergence holds

System Matrices

$$\mathcal{Q}_1^N : W^N \to Z^N$$

The matrix representation of $\mathcal{Q}_1^N$ is given by

$$Q^N = \begin{bmatrix} .5 & 0 & 0 & \cdots & 0 \\ .5 & .5 & 0 & \cdots & 0 \\ 0 & .5 & .5 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & .5 & .5 \end{bmatrix}$$

$$K_{FV}^N = \frac{-v_1}{\Delta x} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

$$w^N(t) = \sum_{k=1}^{N} b_k(t) h_k(x)$$

$$b^N(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_N(t) \end{bmatrix}$$

Approximating System

$$(\Sigma_{CG}^{N}) \qquad \dot{b}^{N}(t) = H_{CG}^{N} b^{N}(t)$$

$$H_{CG}^{N} = K_{FV}^{N}[Q^{N}]^{-1} = \begin{bmatrix} * & 0 & 0 & \cdots & 0 \\ * & * & 0 & \cdots & 0 \\ * & * & * & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$

Observe ... <u>triangular</u> form

M. Kohler and J. Saak, "On GPU acceleration of common solvers for <u>quasi-triangular</u> generalized Lyapunov equations", J. Parallel Computing, ***57*** *(2016), 212–221.*

Goal - "Higher Order" – FE – C_0G Scheme

$$\frac{\partial u(t,x)}{\partial t} = \mu \frac{\partial^2 u(t,x)}{\partial x^2} - v \frac{\partial u(t,x)}{\partial x} \qquad u(t,0) = \eta, \quad \mu \frac{\partial u(t,L)}{\partial x} = 0$$

Finite Element for $P(\mu) = \mu \frac{\partial^2}{\partial x^2}$ Higher Order "DG" and "*hp*" FE for $H(v) = v \frac{\partial}{\partial x}$

$$u^N(t) = \sum_{k=1}^{N} u_k(t) h_k(x)$$ initial data $u(0,x) \equiv \varphi(x)$

a comparison of 4 methods:

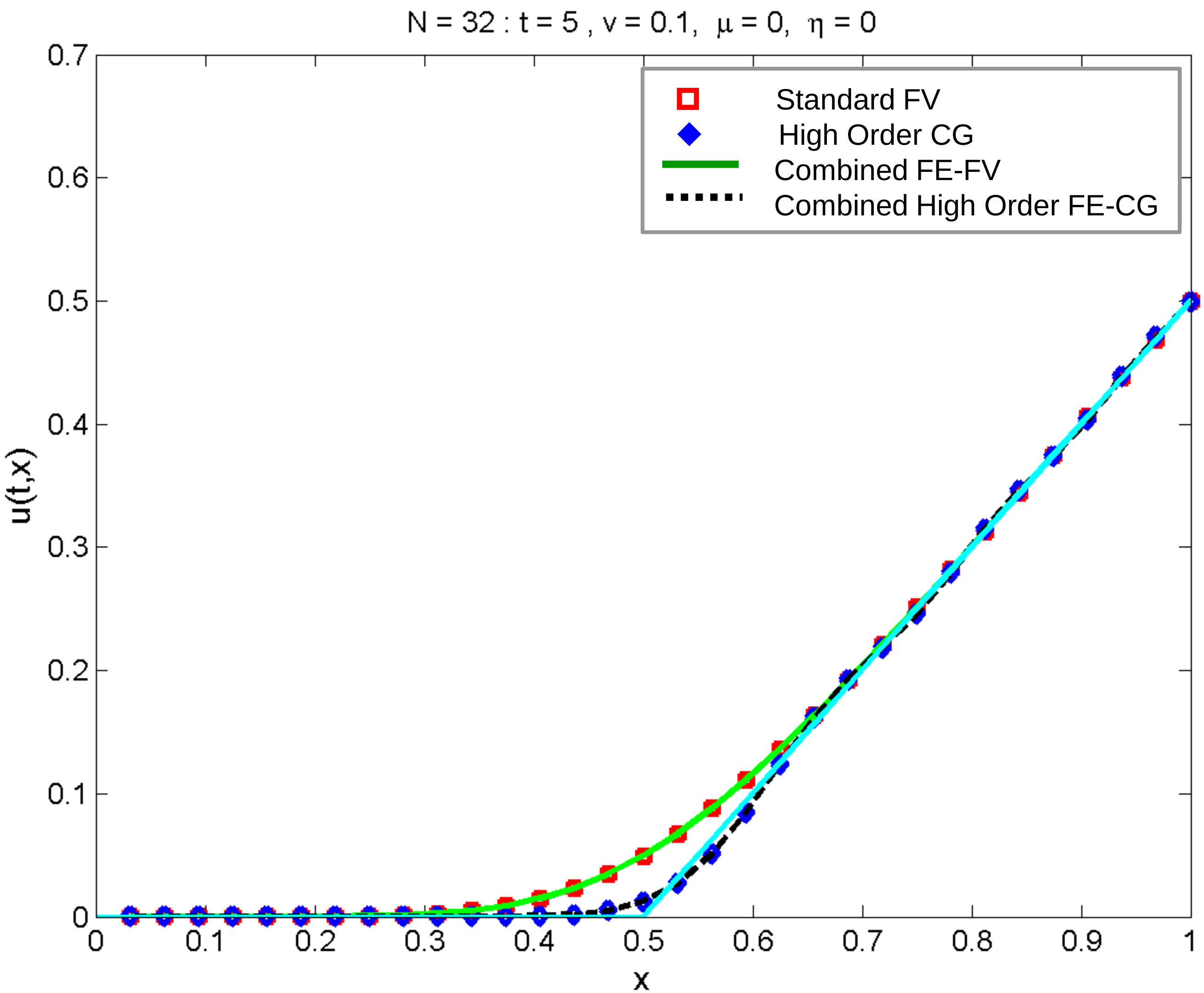
1) "Standard" FV method for the case $0 = \mu$

2) "Higher Order CG" type method for the case $0 = \mu$

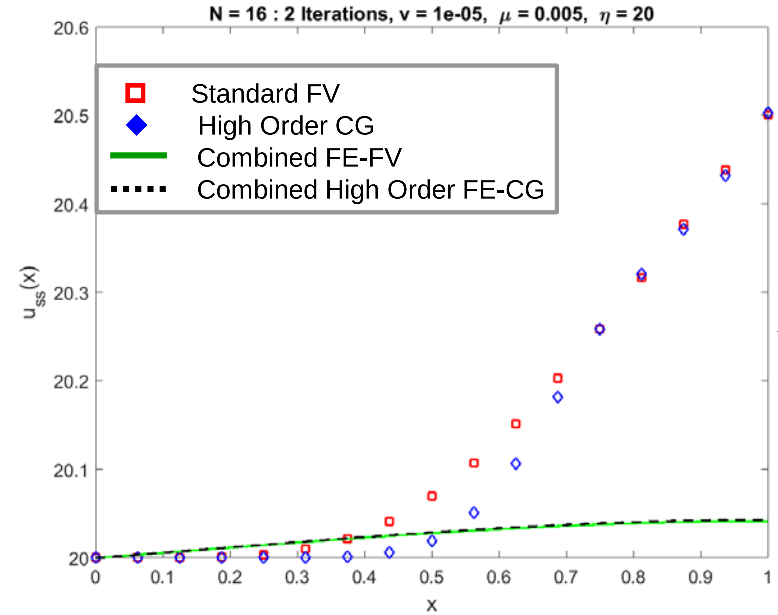
3) FE - FV method for the cases $(i)\ 0 \le \mu < v$ &
$(ii)\ 0 \le v << \mu$ (low flow)

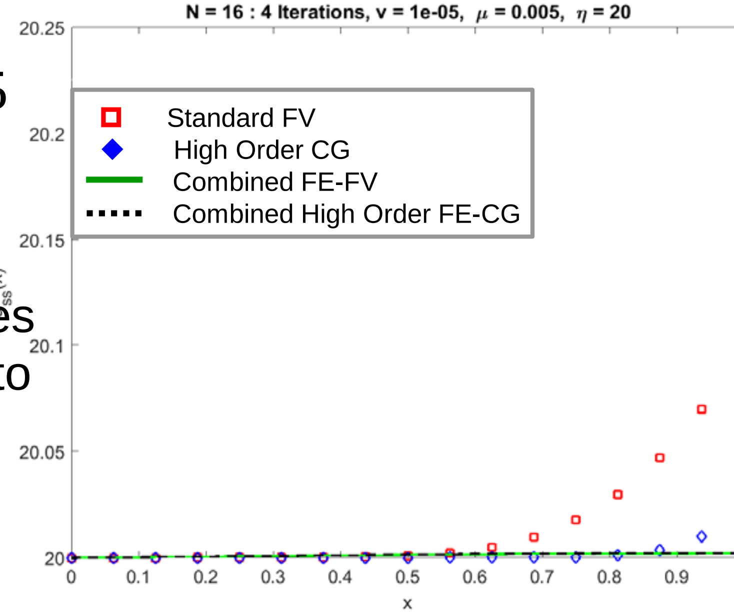
4) FE - CG method for the cases $(i)\ 0 \le \mu < v$ &
$(ii)\ 0 \le v << \mu$ (low flow)

Typical Result



Low Flow Case $v = 0.0001,\ \mu = 0.005$



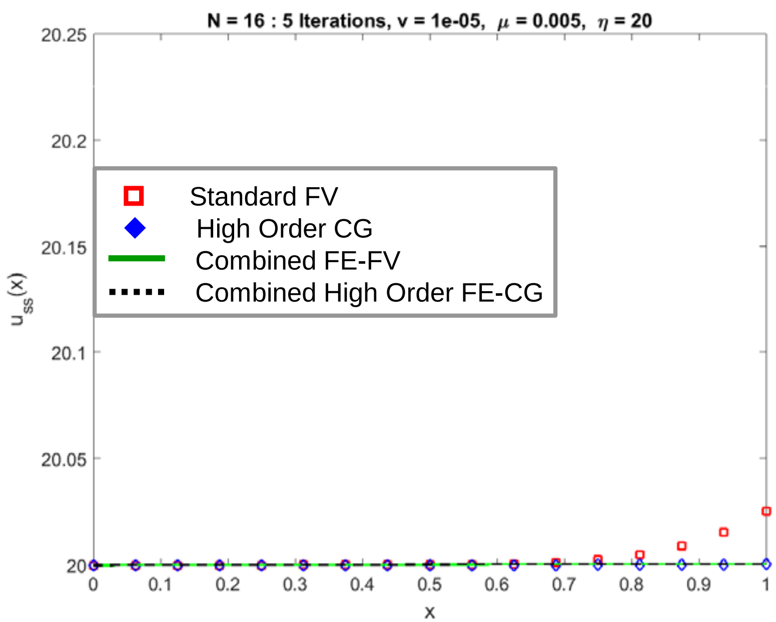


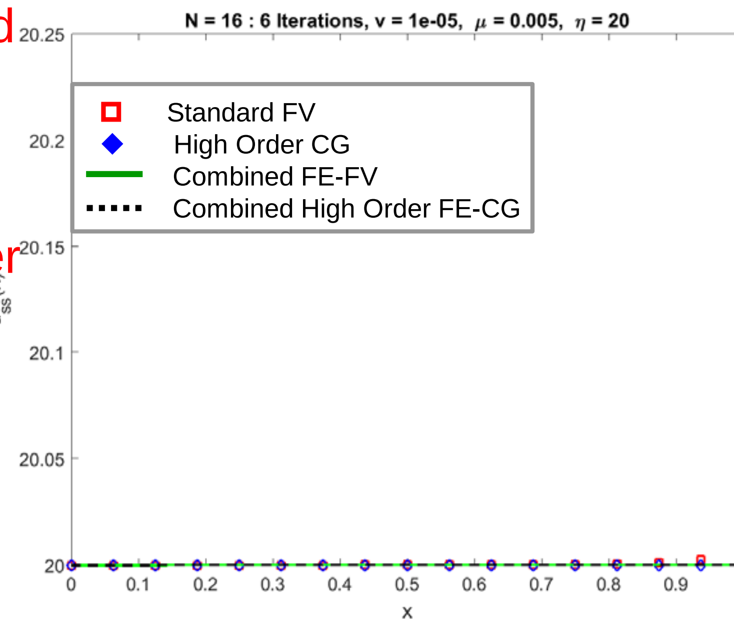
$\mu = 0.005$

$\eta = 20$

All schemes converge to correct solution

BUT Combined FE-FV & FE-C_0G are faster





Time to Converge to Steady State Solution

$\mu = 0.005, \ v = 0.0001$ $L_2 \ error < 10^{-6}$

N Number of cells	Time to converge to Steady State (FV)	Time to converge to Steady State (FE)	Time to converge to Steady State (FE-FV)	Time to converge to Steady State (FE-C_0G)
4	0.1570 sec	0.0516 sec	0.0304 sec	0.0233 sec
16	0.3006 sec	0.1997 sec	0.0957 sec	0.0914 sec
64	1.3659 sec	0.9435 sec	0.4043 sec	0.3150 sec

Note: Nothing special about 1^{st} order splines

Working on higher order methods… *hp* - Finite Element & DG methods

Burns, J. A. and Cheung, J., "*High Order Approximations for Model Reduction in PDE Control Systems*, 2019 ACC, Submitted.

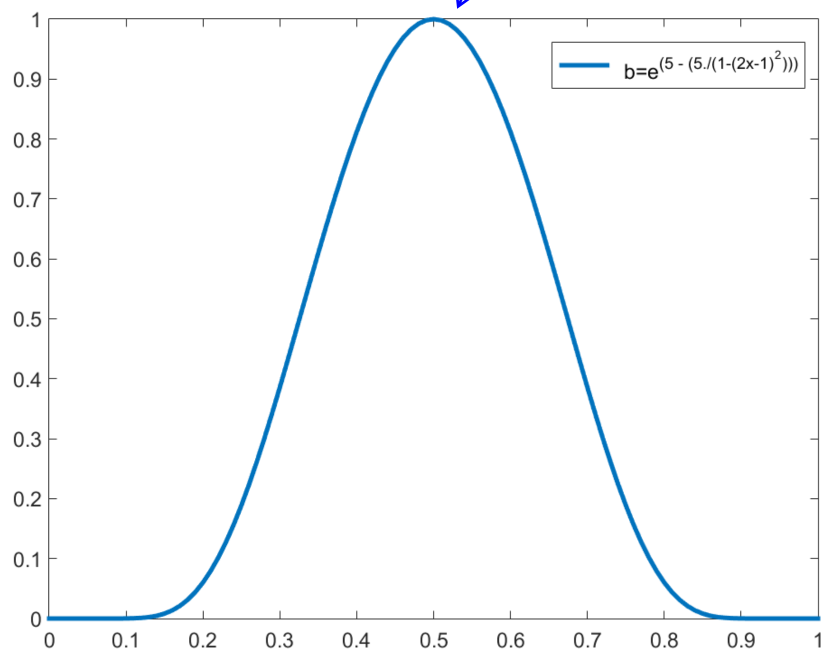
! A natural & rigorous model order reduction method !

hp – FE – C_0G Method: Optimal Error Estimates

$$\frac{\partial z(t,x)}{\partial t} = \mu \frac{\partial^2 z(t,x)}{\partial x^2} - v \frac{\partial z(t,x)}{\partial x} + b(x)u(t) \qquad z(t,0)=0, \quad z(t,1)=0$$

$$J = \int_0^{+\infty} \left\{ \int_0^1 q(x)[z(t,x)]^2 dx + R[u(t)]^2 \right\} dt \qquad u^{opt}(t) = -\int_0^1 k(x) z(t,x) dx$$

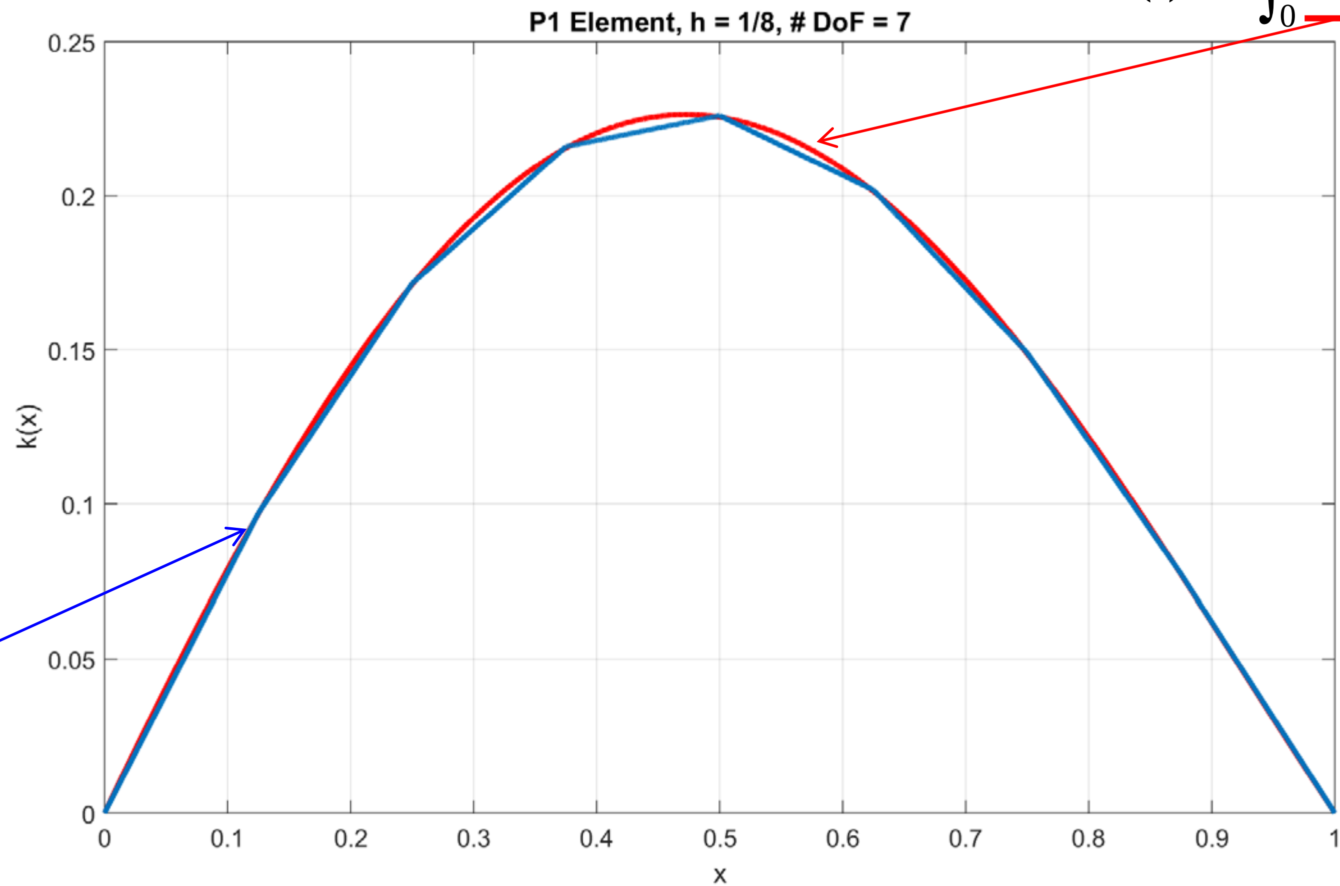
$\mu = 0.01,\ v = 0.1,\ q(x) = 1,\ R = 1,\ \underline{b(x)} = C^\infty$ "bump function"



hp – FE – C_0G Method: Optimal Error Estimates

NON-ADAPTIVE *hp*

$$u^{opt}(t) = -\int_0^1 k(x)\,u(t,x)\,dx$$



7 DOF

FOR ELLIPTIC BVP …

B. Guo and I. Babuška, "The *hp* version of the finite element method", Computational Mechanics, 1 (1986), 21–41.

WITH ADAPTIVE *hp*

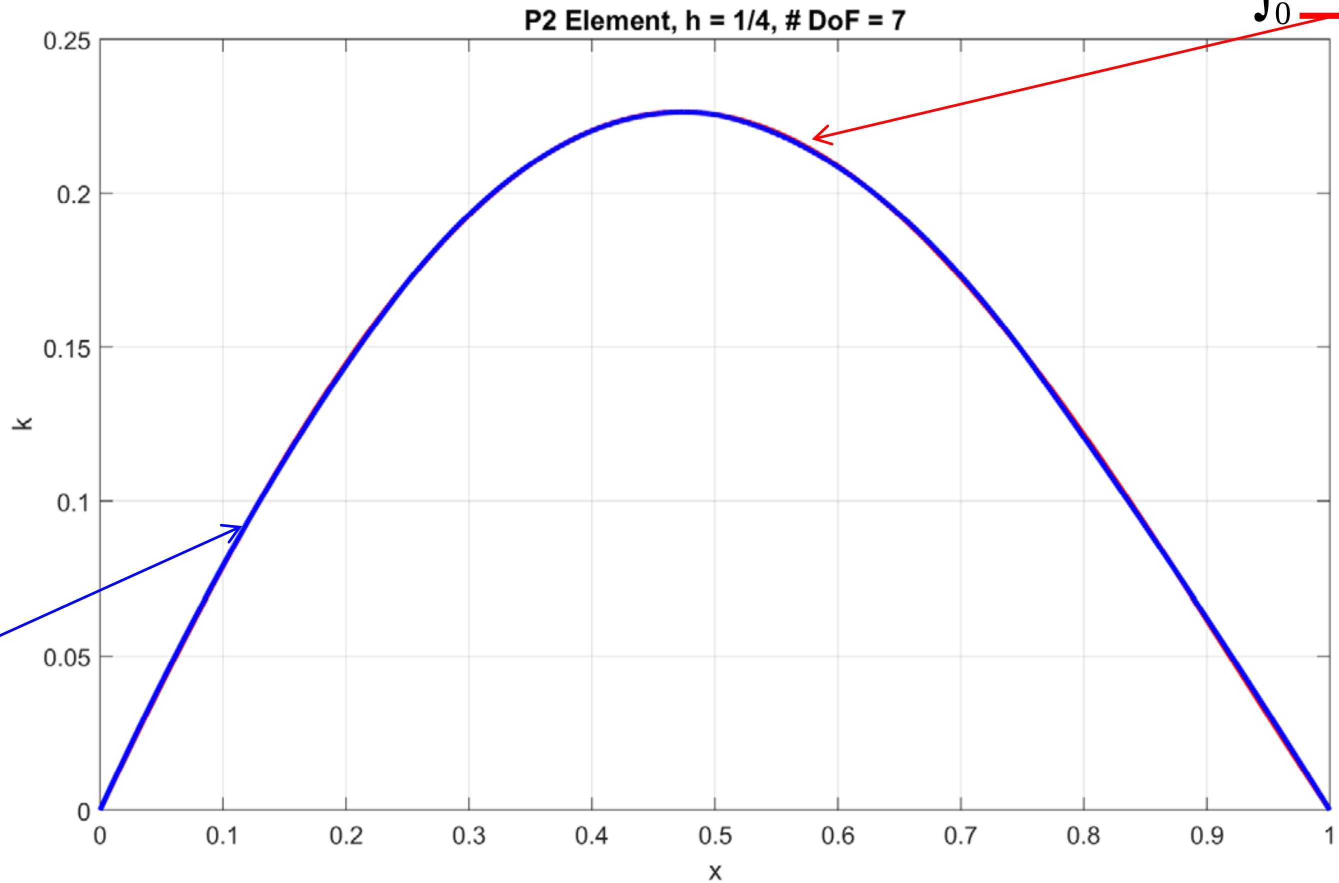
$$\left\| e_{hp} \right\|_{H^1(\Omega)} \le C_1 e^{-C_2 N_{DOF}^{1/3}}$$

hp – FE – C_0G Method: Optimal Error Estimates

NON-ADAPTIVE *hp*

$$u^{opt}(t) = -\int_0^1 k(x)u(t,x)dx$$

7 DOF



FOR RICCATI PDES …

Burns, J. A. and Cheung, J., "*A Convergence Theory for a Galerkin Approximation of Algebraic Riccati Equations*", in preparation.

WITH ADAPTIVE *hp*

$$\left\|e_{hp}\right\|_{H^1(\Omega)} \le C_1 e^{-C_2 N_{DOF}^{1/3}}$$

ICAM

Optimal *hp* Convergence Rates fo *hp* – FE – C_0G

VIRGINIA TECH.

k = 1		
h	Gain Error	Rate
1/2	0.0334	
1/4	0.0081	2.0439
1/8	0.0020	2.0179
1/16	5.0349e-04	1.9900
1/32	1.2583e-04	2.0005
1/64	3.1454e-05	2.0002

63 DOF

k = 2		
h	Gain Error	Rate
1/2	0.0033	
1/4	4.2957e-04	2.9415
1/8	5.4140e-05	2.9881
1/16	6.7802e-06	2.9973
1/32	8.4792e-07	2.9993
1/64	1.0600e-07	2.9999

k = 3		
h	Gain Error	Rate
1/2	2.7947e-04	
1/4	1.8456e-05	3.9205
1/8	1.1731e-06	3.9757
1/16	7.3646e-08	3.9936
1/32	4.6082e-09	3.9983
1/64	6.5729e-10	2.8096

k = 4		
h	Gain Error	Rate
1/2	2.5412e-05	
1/4	8.3996e-07	4.9190
1/8	2.6626e-08	4.9794
1/16	8.4931e-12	11.6143
1/32	1.8790e-10	-4.4675
1/64	1.3663e-09	-2.8622

7 DOF

Valid in 2D and 3D with only C_0 smoothness across element boundaries

! A "natural" & rigorous model order reduction method !

APPLIED TO 2D SENSOR PLACEMENT PROBLEM

WORKING ON 3D DYNAMIC SENSOR PROBLEM

MORE WORK TO BE DONE

THANKS FOR YOUR ATTENTION