Fundamentals

Set Theory

Definition: A set is an undefined concept that represents our intuitive understanding of a collection or aggregate of designated elements or members. It is a many that is a one.

We will usually use upper case letters such as $A, B,$ etc. to denote sets. On the other hand, we will usually use lower case letters such as $x, y,$ etc. to denote members of a set. We have an undefined binary relation between elements and sets that is denoted by

$$x \in A$$

We read this "x is a member of the set A". In axiomatic set theory, there is no distinction between sets and elements. All objects are sets.

Creating Sets:

There are two basic ways to create sets.

Listing the elements:

$\mathbb{N} = \{1,2,3,4,5,6,\ldots\}$ is the infinite set of natural numbers.
$\mathbb{E} = \{2,4,6,8,10,\ldots\}$ is the infinite set of even natural numbers.
$\mathbb{O} = \{1,3,5,7,9,11,\ldots\}$ is the infinite set of odd natural numbers.
$\mathbb{Z} = \{\ldots,-5,-4,-3,-2,-1,0,1,2,3,4,5,\ldots\}$ is the infinite set of integers.
$\{a\}$ is the singleton set that contains the element a.
$\{\ \}$ is the empty set, which is also denoted by $\varnothing$.
$\{a, a\}$ is the same set as $\{a\}$. A set never contains duplicates .

Using Set Builder Notation:

$\mathbb{E} = \{x \in \mathbb{N} \mid x \text{ is even}\} = \{x \in \mathbb{N} \mid (\exists y \in \mathbb{N})(x = 2y)\}$
$\mathbb{O} = \{x \in \mathbb{N} \mid x \text{ is odd}\} = \{x \in \mathbb{N} \mid (\exists y \in \mathbb{N})(x = 2y - 1)\}$
$\mathbb{P} = \{x \in \mathbb{N} \mid x \text{ is prime}\} = \{x \in \mathbb{N} \mid (\forall y \in \mathbb{N})(\forall z \in \mathbb{N})\ (x \geq 2 \wedge (x = y \cdot z \rightarrow (y = 1 \vee z = 1)))$
$\mathbb{Q} = \{x \in \mathbb{R} \mid x \text{ is a rational}\} = \{x \in \mathbb{R} \mid (\exists y \in \mathbb{Z})(\exists z \in \mathbb{Z})\ (z \neq 0 \wedge (x \cdot z = y)\}$

Can a "set" be too big? Yes, it can. Russell's Paradox shows why.

Russell's Paradox:

Consider the "set" $A = \{x \mid x \notin x\}$. Most sets are not members of themselves. Therefore, this "set" must be extraordinarily large. Now,

Case I: Suppose $A \in A$. Then, A must have the property that defines it. That is, $A \notin A$.

Case II: Suppose $A \notin A$. Then, A has the property that defines it. So, $A \in A$.

In summary, we have shown that $A \in A \leftrightarrow A \notin A$. This is indeed a paradox. What is wrong? Put simply, A is too large, and, thus, cannot be a set. This means you cannot use the set builder notation indiscriminately. The elements that are being collected using set builder notation must come from a known set. We will always assume that our elements are coming from some universal set U. Thus, in general, given any predicate $P(x)$, $x \in U$, we may build the following set

$$A = \{x \in U \mid P(x)\}$$

Set Relations:

Definition: If A and B are sets, 'A is a **subset** of B' is the name of the relation $A \subseteq B$ defined by

$$A \subseteq B \cdot \overset{\text{def}}{=} \cdot (\forall x)((x \in A) \rightarrow (x \in B))$$

This says that every element of A is an element of B. If A is a subset of B, then B is a superset of A. Every set is a subset of itself.

Exercises:

(1) Which set has the greater cardinality[6,7] N or E?

(2) Show that Q has the same cardinality as N.

(3) Show that the empty set $\varnothing$ is a subset of every set.

(4) Show that $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$

(5) Decide, among the following sets, which are subsets of which.
   (a) $A = \{x \in \mathrm{P} \mid x^2 - 8x + 12 = 0\}$
   (b) $B = \{2,4,6\}$
   (c) $C = \{2,4,6,8,\ldots\}$
   (d) $D = \{6\}$

(6) List all the subsets of the set $\{-1,0,1\}$.

(7) In each of the following, determine whether the statement is true or false. If it is true, prove it. If it is false, disprove it with an example, called a counterexample.
   (a) If $x \in A$ and $A \in B$, then $x \in B$.
   (b) If $A \subseteq B$ and $B \in C$, then $A \in C$.

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[6] The cardinality of a set is the number of elements in it.

[7] Using set theory, we can develop a theory of infinite sets. This is due the mathematician Georg Cantor (1845-1918).

(c) If $\neg(A \subseteq B)$ and $B \subseteq C$, then $\neg(A \subseteq C)$.
(d) If $\neg(A \subseteq B)$ and $\neg(B \subseteq C)$, then $\neg(A \subseteq C)$.
(e) If $x \in A$ and $\neg(A \subseteq B)$, then $x \notin B$.
(f) If $A \subseteq B$ and $x \notin B$, then $x \notin A$.

Definition: If A and B are sets, 'A is **equal** to B' is the name of the relation $A = B$ defined by

$$A = B \cdot \overset{\text{def}}{=} \cdot (A \subseteq B) \wedge (B \subseteq A)$$

Definition: If A and B are sets, 'A is a **proper subset** of B' is the name of the relation $A \subset B$ defined by

$$A \subset B \cdot \overset{\text{def}}{=} \cdot (A \subseteq B) \wedge (A \neq \mathrm{B})$$

Set Operations:

Definition: If A and B are subsets of U, 'A **union** B' is the name of the set $A \cup B$ defined by

$$A \cup B \cdot \overset{\text{def}}{=} \cdot \{x \in U \mid (x \in A) \vee (x \in B)\}$$

Exercises:

(8) Prove that $A \subseteq B \leftrightarrow A \cup B = B$

(9) Prove that $A \subseteq B \leftrightarrow A \cap B = A$

Definition: If A and B are subsets of U, 'A **intersect** B' is the name of the set $A \cap B$ defined by

$$A \cap B \cdot \overset{\text{def}}{=} \cdot \{x \in U \mid (x \in A) \wedge (x \in B)\}$$

Examples:

(1) If $A = \{1,2,3,4\}$ and $B = \{3,4,5\}$, then $A \cup B = \{1,2,3,4,5\}$.
(2) If $A = \{1,2,3,4\}$ and $B = \{3,4,5\}$, then $A \cap B = \{3,4\}$.
(3) If **Im** denote the set of imaginary numbers, then $\mathrm{P} \cap \mathbf{Im} = \varnothing$.

Definition: If A is a subset of U, the '**complement** of A in U' is the name of the set A' defined by

$$A' \cdot \overset{\text{def}}{=} \cdot \{x \in U \mid x \notin A\}$$

Definition: If A and B are subsets of U, the '**relative complement** of B in A' is the name of the set $A - B$ defined by

$$A - B \overset{\text{def}}{\cdot=\cdot} \{x \in U \mid (x \in A) \wedge (x \notin B)\}$$

Definition: If A is a subset of U, the "**power set** of A" is the name of the set $\mathcal{P}(A)$ defined by

$$\mathcal{P}(A) \overset{\text{def}}{\cdot=\cdot} \{x \mid x \subseteq A\}$$

Exercises:

(10) Prove DeMorgan's Laws for sets:

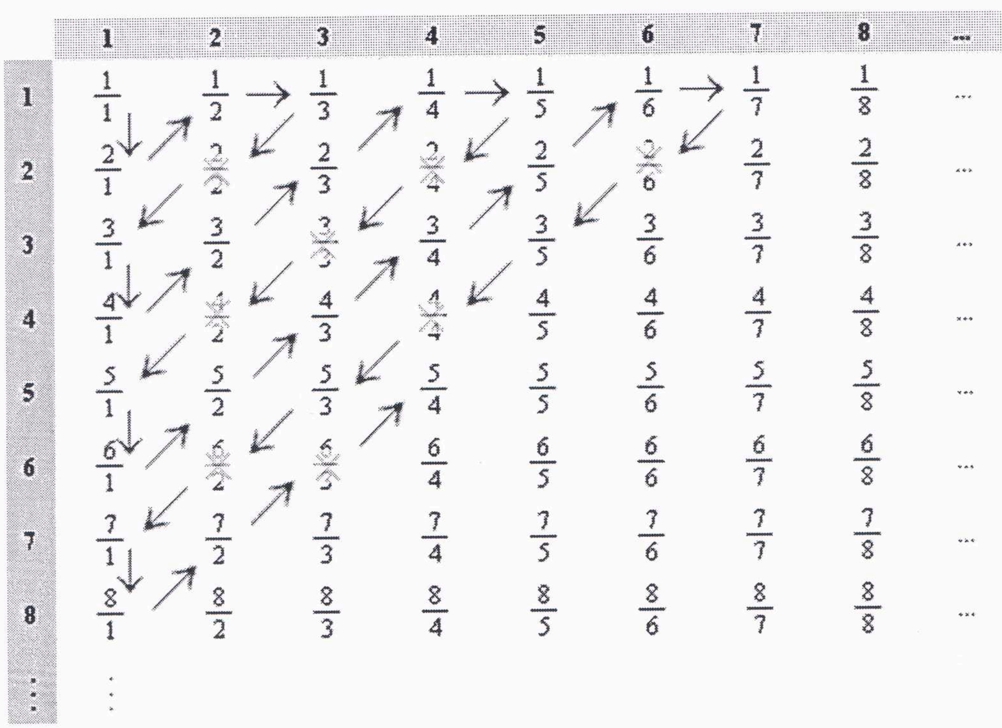
(a) $(A \cup B)' = A' \cap B'$

(b) $(A \cap B)' = A' \cup B'$

(11) If $A = \{a,b,c,d\}$, determine $\mathcal{P}(A)$.

(12) If $A = \{3,6,9,12,15,18,21,\ldots\}$ and $B = \{5,10,15,20,25,30,35,\ldots\}$, find $A \cap B$.

Theorem: The rational numbers are countable



Theorem: The real numbers are uncountable

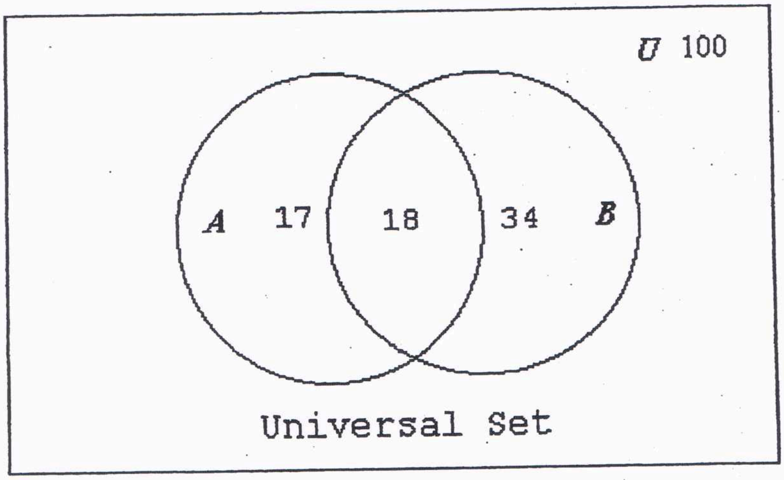
$$
\begin{array}{l}
s_1 = 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\ldots \\
s_2 = 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\ldots \\
s_3 = 0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\ldots \\
s_4 = 1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\ldots \\
s_5 = 1\,1\,0\,1\,0\,1\,1\,0\,1\,0\,1\ldots \\
s_6 = 0\,0\,1\,1\,0\,1\,1\,0\,1\,1\,0\ldots \\
s_7 = 1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\ldots \\
s_8 = 0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,1\ldots \\
s_9 = 1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0\ldots \\
s_{10} = 1\,1\,0\,1\,1\,1\,0\,0\,1\,0\,1\ldots \\
s_{11} = 1\,1\,0\,1\,0\,1\,0\,0\,1\,0\,0\ldots \\
\vdots \qquad \vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\ddots
\end{array}
$$

$$
s = 1\,0\,1\,1\,1\,0\,1\,0\,0\,1\,1\ldots
$$

Proof: In particular, we claim that the unit interval $[0,1]$ is uncoutable. We prove this by contradiction. Assume that $[0,1]$ is countable. So, we can list these numbers, in binary, as illustrated above. But, then, we can produce a number that is not in our list, as shown above. $\Rightarrow\Leftarrow$

Venn Diagrams

Example: In a survey of 100 college students, 35 were registered in College Algebra, 52 were registered in Introduction to Computer Science, and 18 were in both courses. How many were registered in neither course?



First, We let A = Set of students in College Algebra

B = Set of students in Introduction to Computer Science

We use the notation $n(A)$ to represent the number of elements in A.

This information tells us that

$n(A) = 35$ $n(B) = 52$ $n(A \cap B) = 18$ $n(U) = 100$. So,

1. $n(A - A \cap B) = n(A) - n(A \cap B) = 35 - 18 = 17$
2. $n(B - A \cap B) = n(B) - n(A \cap B) = 52 - 18 = 34$
3. $n((A \cup B)') = n(U) - n(A \cup B) = 100 - (17 + 18 + 34) = 100 - 69 = 31$

That is, $n(\text{students not regestered in either course}) \equiv n((A \cup B)') = 31$.

Exercise1: In a student survey of 500 people, 200 indicated that they would be buying a home theatre system within the next month; 150 indicated they would buy a car, and 25 said they would purchase both a home theatre system and a car. How many will purchase neither? How many will purchase only a car? Use a Venn diagram to determine the solution.

Exercise2: In a consumer survey, 200 indicated that they would attend Summer Session I and 150 indicated they would attend Summer Session II. If 75 students plan to attend both summer sessions and 275 indicated that they would attend neither session, how many students participated in the survey?
Use a Venn diagram to determine the solution.

Exercise3: This question refers to figure 1.

1. How many elements are in set *A*?
2. How many elements are in set *B*?
3. How many elements are in *A* or *B*?
4. How many elements are in *A* and *B*?
5. How many elements are in *A* but not in *C*?
6. How many elements are not in *A*?
7. How many elements are in *A* and *B* and *C*?
8. How many elements are in *A* or *B* or *C*?

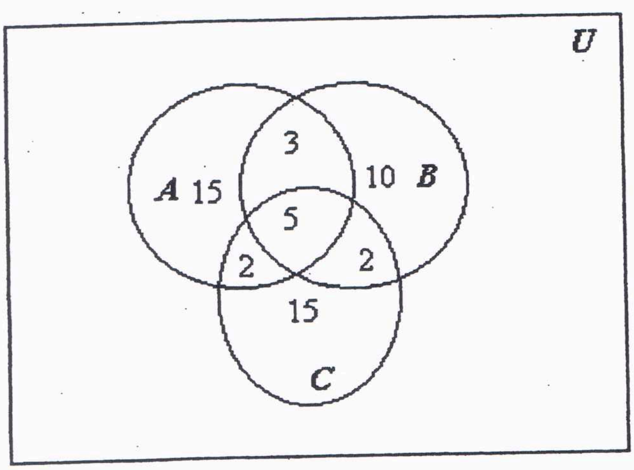


Figure 1

Venn Diagram Problems

1. In a group of students registering for math
   a) 20 registered because they found math fun
   b) 28 registered because they found math challenging
   c) 32 registered because they found math useful
   d) 11 registered because they found math challenging and useful
   e) 2 registered because they found math fun, challenging, and useful
   f) 8 registered because they found math fun and either useful or challenging
   g) 7 registered because they found math fun and useful

   a) Completely label and fill in a three-set Venn diagram for the above scenario.
   b) How many students registered because they
      i) just found math challenging?
      ii) just found math fun?
      iii) just found math useful?

2. There are 120 students from which
   - 54 listen to Rock
   - 40 listen to Country
   - 40 listen to Alternative
   - 12 listen to Country & Rock
   - 10 listen to Country & Alternative
   - 16 listen to Rock & Alternative
   - 7 listen to all 3

   a) Completely label and fill in a three-set Venn diagram of the above scenario.
   b) How many listen to only Rock & Country?
   c) How many don't listen to any of the three?

3. From 100 pet owners
   - 56 owned fish
   - 35 owned dogs
   - 29 owned cats
   - 12 owned none of the above
   - 14 owned dogs & fish.
   - No one owned all three.
   - 16 owned cats & fish.

   a) How many people owned cats & dogs?
   b) How many people owned just fish?

4. In a survey of children who saw three different shows at Walt Disney World, the following information was gathered:

- 39 children liked *The Little Mermaid*
- 43 children liked *101 Dalmatians*
- 56 children liked *Mickey Mouse*
- 7 children liked *The Little Mermaid* and *101 Dalmatians*
- 10 children liked *The Little Mermaid* and *Mickey Mouse*
- 16 children liked *101 Dalmatians* and *Mickey Mouse*
- 4 children liked *The Little Mermaid, 101 Dalmatians*, and *Mickey Mouse*
- 6 children did not like any of the shows

a) How many students were surveyed?
b) How many liked *The Little Mermaid* only?
c) How many liked *101 Dalmatians* only?
d) How many liked *Mickey Mouse* only?

5. In a certain group of football players:

- 60 played for Packers
- 40 played for Lions
- 35 played for Bears
- 20 played for Packers & Lions
- 15 played for Lions & Bears
- 12 played for Packers & Bears
- 3 played for all three
- 3 didn't play for any of the three

a) How many players are there altogether in this group?

At the site http://www.rwc.uc.edu/koehler/comath/java/venn.html, there are the following instructions for manipulating a java applet on Venn diagrams. Use this applet to practice the various set operations. Please note that "+" is used for "$\cup$", "&" for "$\cap$", and A' for $\sim A$.

The applet below creates problems for you to practice using Venn diagrams. You will be given a set theoretical expression in three sets, and asked to color in the Venn diagram to correctly represent the expression. When you have computed your answer, click in the various regions of the diagram to "color" in the diagram, and then click on the "Check Answer" button to find out if you did the problem correctly. The "Start Over" button will clear the diagram and the "Next Problem" button will generate another expression for you to practice with.