Departamento de Matemáticas
Recinto de Río Piedras

Topology Qualifying Exam — August 2016

Do exactly **five** *of the following problems. In order to obtain credit you must show all your work.*
The passing grade at the M.S. level is 2/3 and at the Ph.D. level is 3/4.

Problem 1. (*20 points*) A topological space X is called *locally Euclidean* if for any $x \in X$, there is an open neighborhood U_x which is homeomorphic to the Euclidean space $\mathbb{R}^n$. Prove that if X is a connected locally Euclidean space, then X is path connected.

Problem 2. (*20 points*) Prove the following result directly (without using any theorem from covering spaces). Let $f : [0,1] \to S^1 = \{z \in \mathbb{C};\ |z| = 1\}$ be a continuous function with $f(0) = 1$. Then there is a unique continuous function $\tilde{f} : [0,1] \to \mathbb{R}$ such that $\tilde{f}(0) = 0$ and $f(t) = e^{i\tilde{f}(t)}$.

Problem 3. (*20 points*) Suppose that X is a compact metric space, and $\cup_{i=1}^{n} U_i$ is a finite open cover of X. Prove that there exist n open subsets $\{V_i\}_{i=1}^{n}$ such that $V_i \subset \overline{V_i} \subset U_i$ and $\cup_{i=1}^{n} V_i$ is also an open cover of X.

Problem 4. Let $X = [-1,1]$ and $\mathscr{T} = \{U \in \mathscr{P}(X) \mid 0 \notin U \text{ or } (-1,1) \subseteq U\}$.

(i) (*6 points*) Show that $(X, \mathscr{T})$ is a topological space.

(ii) (*7 points*) Show that $(X, \mathscr{T})$ is not Hausdorff.

(iii) (*7 points*) Show that $(X, \mathscr{T})$ is first countable.

Problem 5. Let $(\mathbb{R}, \mathscr{T}_0)$ be the usual euclidean space. Define a new topology on $\mathbb{R}$ by letting $\mathscr{T}_1 = \{U \in \mathscr{P}(\mathbb{R}) \mid U = \varnothing \text{ or } \mathbb{R} - U \text{ is compact in } (\mathbb{R}, \mathscr{T}_0)\}$

(i) (*4 points*) Show that $\mathscr{T}_1 \subseteq \mathscr{T}_0$.

(ii) (*8 points*) Show that $(\mathbb{R}, \mathscr{T}_1)$ is compact.

(iii) (*8 points*) Show that $(\mathbb{R}, \mathscr{T}_1)$ is separable.

Problem 6. (*20 points*) Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a function between compact Hausdorff spaces. Let

$$G_f = \{(x,y) \in X \times Y \mid y = f(x)\},$$

be the graph of f. Show that if G_f is closed in $X \times Y$, then f is continuous.

Problem 7. (*20 points*) Let $(X, \mathcal{T}_X)$ be a path-connected topological space. Show that $Y = \prod_{n=1}^{\infty} X_n$ where $X_n = X$ for all n, with the product topology, is also path-connected.