Departamento de Matemáticas
Recinto de Río Piedras

Topology Qualifying Exam — February 2012

Do exactly **five** *of the following problems. In order to obtain credit you must show all your work.*

1. Let X be a set. Recall that a topology on X is the collection $\mathscr{T}$ of subsets of X with conditions:

   (i) $\varnothing \in \mathscr{T}$ and $X \in \mathscr{T}$;
   (ii) If $U_\alpha \in \mathscr{T}$ for $\alpha \in I$, then $\cup_{\alpha \in I} U_\alpha \in \mathscr{T}$;
   (iii) If $U_1, U_2, \cdots, U_k \in \mathscr{T}$, then $\cap_{i=1}^{k} U_i \in \mathscr{T}$.

   (a) Let $\{\mathscr{T}_\gamma\}_{\gamma\in\Gamma}$ be collection of topologies $\mathscr{T}_\gamma$ on X. Prove that $\cap_{\gamma\in\Gamma}\mathscr{T}_\gamma$ is a topology on X.
   (b) Construct an example of $\{\mathscr{T}_\gamma\}_{\gamma\in\Gamma}$ such that each $\mathscr{T}_\gamma$ is a topology on X but $\cup_{\gamma\in\Gamma}\mathscr{T}_\gamma$ is not a topology on X.

2. Let X be a set and $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ be a function defined on the set of subsets of X, $\mathscr{P}(X)$ that satisfies the following conditions:

   (i) $\forall A \in \mathscr{P}(X),\ A \subseteq cl(A)$.
   (ii) $\forall A \in \mathscr{P}(X),\ cl(cl(A)) = cl(A)$.
   (iii) $cl(\varnothing) = \varnothing$.
   (iv) If $A, B \in \mathscr{P}(X)$, then
   $$cl(A \cup B) = cl(A) \cup cl(B).$$

   Show that the collection

   $$\mathscr{T} = \{U \in \mathscr{P}(X) \mid \text{there is a subset } C \text{ of } X \text{ such that } cl(C) = C \text{ and } U = X - C\ \}$$

   is a topology over X.

3. A topological space X is called *locally Euclidean* if for any $x \in X$, there is an open neighborhood U_x which is homeomorphic to the Euclidean space $\mathbb{R}^n$.

   (a) Prove that if X is connected locally Euclidean space, then X is path connected.
   (b) Construct a non-Hausdorff space which is locally Euclidean.

4. A continuous map $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is said to be **final**, if for each topological space $(Z, \mathcal{T}_Z)$ each set-function $g : (Y, \mathcal{T}_Y) \to (Z, \mathcal{T}_Z)$ is continuous whenever $g \circ f : (X, \mathcal{T}_X) \to (Z, \mathcal{T}_Z)$ is continuous.

   (a) Show that the composition of final maps is final.

   (b) Consider the functions $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ and $h : (Y, \mathcal{T}_Y) \to (W, \mathcal{T}_W)$. Show that if the composition $h \circ f$ is final, then h is final.

5. Let X be a compact metric space and $A \subset X$ be a closed subset such that $X - A$ is countable. Prove that there is a continuous map $f : X \to A$ such that $f|_A$ is the identity map on A.

6. Let $(X, \mathcal{T}_X)$ be a topological space and $(Y, \mathcal{T}_Y)$ be a Hausdorff topological space. Suppose that $f, g : X \to Y$ are continuous functions.

   (a) Suppose that $A \subseteq X$ is non empty and that $f(a) = g(a)$ for all $a \in A$. Show that $f(x) = g(x)$ for all $x \in \overline{A}$.

   (b) Let $B = \{x \in X \mid f(x) = g(x)\}$. Show that B is closed.

7. Let I be an index set. For each $\alpha \in I$, X_α is a compact metric space with at least two points.

   (a) Prove that if I is countable, then $\prod_{\alpha \in I} X_\alpha$ is a compact metric space. (You do not need to prove compactness which follows from Tychonoff's Theorem.)

   (b) Prove that if I is not countable, then $\prod_{\alpha \in I} X_\alpha$ is not a metric space.