Departamento de Matemáticas
Recinto de Río Piedras

Topology Qualifying Exam — March 2011

Do exactly **five** *of the following problems. In order to obtain credit you must show all your work.*

Problem 1 In a topological space X, a closed subset $F \subset X$ is called *prime* if whenever G and H are closed sets with $F = G \cup H$, then $F = G$ or $F = H$.

(a) Prove that if X is a space of finitely many points, then every prime (closed) subset F of X is the closure of a single point.

(b) Construct a topological space X with a prime (closed) subset which is not the closure of any single point set.

Problem 2 Let (X, d) be the Euclidean plane with the usual metric; let $\mathbf{0} = (0, 0)$ be the origin in this plane. Define a new metric on $X = \mathbb{R}^2$ as follows:

$$\tilde{d}(p,q) = \begin{cases} \mathbf{0}, & \text{if } p = q \\ d(p,q), & \text{if } p \neq q \text{ and the line through } p \text{ and } q \text{ passes through } \mathbf{0} \\ d(p,\mathbf{0}) + d(q,\mathbf{0}), & \text{otherwise} \end{cases}$$

(a) Let $p = (1, 1)$. Describe (you may use a diagram) each of the following ϵ-balls; $B_{\tilde{d}}(p; 1/2)$, $B_{\tilde{d}}(p, 2)$ and $B_{\tilde{d}}(\mathbf{0}, 1/2)$.

(b) Show that $(X, \tilde{d})$ is not separable.

Problem 3 A topological space X is called *locally Euclidean* if for any $x \in X$, there is an open neighborhood U_x of x which is homeomorphic to the Euclidean space $\mathbb{R}^n$.

(a) Prove that if X is locally Euclidean space, then X is a T_1-space, that is, for any two points $x, y \in X$, there are two open sets $U_x \ni x$, and $U_y \ni y$, such that $y \notin U_x$ and $x \notin U_y$.

(b) Construct a non-Hausdorff space which is locally Euclidean.

Problem 4 Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is *strongly continuous*, if $f(\overline{A}) \subseteq f(A)$ for all $A \subseteq X$. Show the following proposition

"*f is strongly continuous $\iff$ $f^{-1}(B)$ is closed for all $B \subseteq Y$.*"

Problem 5 A topological space X is said to be sequentially compact if every sequence $\{x_n\} \subset X$ has a convergent subsequence $\{x_{n_k}\}$.

(a) Construct a space X which is sequentially compact but not compact.

(b) Construct a space Y which is compact but not sequentially compact.

Problem 6 Let A, B be two non-empty subsets of $\mathbb{R}$, with the usual topology. Define

$$C = \{x + y : x \in A \text{ and } y \in B\}.$$

(a) Show that, if A or B is open, then C is open.

(b) Show that, if A and B are compact, then C is compact.

Problem 7 Prove that any second countable compact Hausdorff space X is homeomorphic to a metric space. (X is called *second countable* if there is a countable base of open sets for the topology.)