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Examen Graduado de Aprovechamiento: Topología
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Escoja exactamente **tres** de los siguientes cinco problemas.
Para obtener crédito muestre todo su trabajo. Explique claramente su contestación.

1. Let $X = \mathbb{Z}^+$ be the set of positive integers,

$$U_a(b) = \{b + na \mid n \in \mathbb{Z}\} \cap X \quad \text{y} \quad \mathfrak{B} = \{U_a(b) \mid a, b \in X, (a, b) = 1\}.$$

   Show that $\mathfrak{B}$ is a base for a topology over X
   (**Note**. $(a, b) = 1$, means that a and b are relatively prime).

2. Let $C[0, 1]$ be the colection of continuous functions defined over the interval $[0, 1]$. Show that the function $d : C[0, 1] \times C[0, 1] \to [0, +\infty)$ defined by:

$$d(f, g) = \int_0^1 |f(x) - g(x)|\, dx \quad \forall f, g \in C[0, 1]$$

   is a metric.

3. A topological space $(X, \mathcal{T}_X)$ is **locally path connected**, if for each point $x \in X$, there exists a path connected open neighborhood U of x in X. Prove that if a connected space $(X, \mathcal{T}_X)$ is locally path connected, then it is path connected.

4. Let X be the Möbius band, defined as the quotient (or identification) space $([0, \pi] \times [-1, 1])/\sim$, where $\sim$ is the relation that identifies the point $(0, t)$ of $[0, \pi] \times [-1, 1]$ with the point $(\pi, -t)$ for all $t \in [-1, 1]$. Let Y be the space obtained from an annulus by identifying each pair of opposite points of the inner circle. Show that X is homeomorphic to Y.

5. Let $(X, \mathcal{T}_X)$ be a topological space. A sequence (x_n) of points in X **frequents** a set A, if $\forall i \in \mathbb{N}$, there is a $j \in \mathbb{N}$ such that $j \geq i$ y $x_j \in A$. A sequence (x_n) **accumulates** in $x \in X$, if for every open neighborhood U of x, (x_n) frequents U. $x \in X$ is an **accumulation point** of X, if there is a sequence of points in X that accumulates in x.

   (a) Show that if (x_n) is a sequence that converges to $x \in X$, then (x_n) accumulates in x.

   (b) In $(\mathbb{R}, \mathcal{T}_{\mathcal{E}^1})$, find a sequence (x_n) and a point (or several) $x \in \mathbb{R}$ such that (x_n) accumulates in x, but (x_n) does not converge to x.
   (**Note**. Here $\mathcal{T}_{\mathcal{E}^1}$ is the usual topology over $\mathbb{R}$).