Universidad de Puerto Rico
Departamento de Matemáticas
Facultad de Ciencias Naturales
Recinto de Río Piedras

September 2010

PhD Qualifying Exam: Analysis

3.5 hours

You may solve all eight (8) Problems (each worths 20 points) but only the best five (5) solutions will be counted as your grade. The passing grade is 70 points.

1. Let $\alpha > 1$ be arbitrary. Show that the equation $\alpha - z - e^{-z} = 0$ has exactly one solution in the half plane $\{z \in \mathbb{C} : \ \mathrm{Re}(z) > 0\}$, and moreover this solution is real.

2. Compute the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\, dx$ for $a > 0$, by using residue theorem.

3. Let f be a continuous function on $[0, 1]$ with a continuous derivative $f'(x)$. Given $\varepsilon > 0$, prove that there is a polynomial $p(x)$ so that

$$\|f(x) - p(x)\|_\infty + \|f'(x) - p'(x)\|_\infty < \varepsilon.$$

4. (a) Give an example of a sequence (f_n) in $L^1[0, 1]$ such that $\lim_{n\to\infty} \|f_n\|_{L^1[0,1]} = 0$ but (f_n) does not converge to 0 almost everywhere.

(b) Show that if a sequence (f_k) in $L^1[0, 1]$ satisfies $\|f_k\|_{L^1[0,1]} \le 2^{-k}$ for $k \ge 1$, then $f_k \to 0$ almost everywhere.

5. Let $\{X_\alpha\}$ be a collection of mutually disjoint measurable subsets of $\mathbb{R}$. Prove that there are at most countable of them with **nonzero** measure.

6. Let f be a Lebesgue integrable function defined on $[0, 1]$. Prove that

$\displaystyle\lim_{n\to\infty} \int_0^1 f(x) \sin(nx)\, dx = 0.$

7. Let f be an analytic function defined on the unit disc $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ and let M be a positive number. Assume that for every sequence $z_n \in \mathbb{D}$ with $\lim_{n\to\infty} z_n = z_0$, where $z_0 \in \partial\mathbb{D} := \{z \in \mathbb{C} : \ |z| = 1\}$, one has $\liminf_{n\to\infty} |f(z_n)| \le M$. Prove that $|f(z)| \le M$ for each $z \in \mathbb{D}$.

8. Construct a measurable function f on $[0, 1]$ such that $\displaystyle\int_0^1 |f(x)|\ dx < +\infty$, but for any $0 \le c < d \le 1$ one has $\displaystyle\int_c^d (f(x))^2\ dx = +\infty$.