PhD Qualifying Exam: Analysis

You may solve all seven (7) Problems but only the best five (5) solutions will be counted as your grade.

1. Let $x_1 \le x_2 \le x_3 \le \cdots$ be a sequence of positive integers. A term x_n is called **good** if it can be written as sum of previous terms $x_1, x_2, \cdots, x_{n-1}$ (any of them can be repeated). Prove that there are at most finite terms which are not good.

   Hint: you can use the following well-known result from number theory.

   Theorem: For any positive integers $a_1, a_2, \cdots, a_n$, if d is the greatest common factor of $a_1, a_2, \cdots, a_n$, then for any $N \ge a_1 a_2 \cdots a_n$ with d divides N, there are nonnegative integers $b_1, b_2, \cdots, b_n$ such that $N = a_1 b_1 + a_2 b_2 + \cdots a_n b_n$.

2. For this problem assume that $q > 1$ and let N_q be the unique integer satisfying $N_q \le \frac{3}{2 \ln q} < N_q + 1$.

   (a) Show that
   $$\sum_{n=1}^{\infty} n^{3/2} q^{-n} \ge \int_0^{N_q} x^{3/2} q^{-x}\, dx.$$
   Hint: find where the maximum of $f(x) = x^{2/3} q^{-x}$ is attained.

   (b) Show that
   $$\int_0^{N_q} x^{3/2} q^{-x}\, dx \ge \frac{1}{(\ln q)^{5/2}} \int_0^1 t^{3/2} e^{-t}\, dt.$$

3. Show that the integral $\displaystyle\int_0^{\infty} \frac{\ln(x)}{(1+x^2)^2}\, dx$ converges and compute its value. **Justify each step**.

4. (a) Give a precise statement of the Cauchy integral formula.

   (b) Let $D = \{z \in \mathbb{C} : |z| < 1\}$, and suppose that $f : D \to \mathbb{C}$ is analytic with $M := \sup_{z \in D} |f(z)| < \infty$. prove that for $0 < \delta < 1$,
   $$\sup_{|z|<1-\delta} |f'(z)| \le \frac{M}{\delta}.$$

   (c) Show that if $\delta = \frac{1}{n}$ and $f(z) = z^n$, then
   $$\sup_{|z|<1-\delta} |f'(z)| \ge \frac{c_n}{\delta},$$
   where $c_n \to e^{-1}$ as $n \to \infty$.

5. Let $[x]$ be the greatest integer which does not exceed x. Let $g(x) = (-1)^{[x]}$.

   (a) Let f be a continuous function on $[0,1]$. Prove that $\displaystyle\lim_{n\to\infty} \int_0^1 f(x) g(nx)\, dx = 0$.

   (b) Let f be a Lebesgue integrable function on $[0,1]$. Prove that $\displaystyle\lim_{n\to\infty} \int_0^1 f(x) g(nx)\, dx = 0$.

6. Let $f:\ (-1,1) \to \mathbb{R}$ be a nonconstant analytic function, that is, for any $x_0 \in (-1,1)$, the Taylor expansion

$$f(x_0) + f'(x_0)(x - x_0) + \frac{f'(x_0)}{2!}(x - x_0)^2 + \cdots$$

converges to $f(x)$ for $x \in (x_0 - a, x_0 + a)$, where $a = \min\{|x-1|, |x+1|\}$. Suppose $x_1, x_2, \cdots, x_n, \cdots \in [0,1)$ is a sequence with $f(x_n) = 0$. Prove that $\lim_{n\to\infty} x_n = 1$.

7. Let $u:\ \mathbb{R}^2 \to \mathbb{R}$ be a nonconstant smooth function (that is u is at least twice continuously differentiable) with

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for all} \quad (x,y) \in \mathbb{R}^2.$$

Prove that for any $(x_0, y_0) \in \mathbb{R}^2$, there is $(x_1, y_1) \in \mathbb{R}^2$ with $u(x_1, y_1) > u(x_0, y_0)$.