February 9, 2007

PhD Qualifying Exam: Analysis

You may solve all six (6) Problems but only the best five (5) solutions will be counted as your grade.

1. Use contour integration (justify each step) to show that if $a > 0$, then
$$\int_0^\infty \frac{\log(x)}{x^2 + a^2}\, dx = \frac{\pi}{2a}\log(a).$$

2. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is convergent. Prove that the series $\sum_{n=1}^{\infty} \frac{n}{a_1 + a_2 + \cdots + a_n}$ is convergent.

3. Let $W := \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0\}$. Let $s > 0$ and let $f : W \to \mathbb{C}$ be a continuous function such that $\lim_{\substack{|z|\to\infty \\ z\in W}} z f(z) = 0$ and the integral $I_s := \int_{-\infty}^{\infty} \exp(ist) f(t)\, dt$ exists. Prove that
$$\lim_{r\to\infty} \int_{\gamma_r} \exp(isz) f(z)\, dz = I_s,$$
where γ_r is the positively oriented boundary of the set $\{z \in W : |z| < r\}$.

4. (a) Let $g : [a, b] \to \mathbb{R}$ be Lebesgue integrable on $[a, b]$. Show that if $c \in (a, b)$ and $g(x) \leq g(y)$ for every $x \in [a, c]$ and $y \in [c, b]$, then
$$\frac{1}{c-a}\int_a^c g(t)\, dt \leq \frac{1}{b-c}\int_c^b g(t)\, dt.$$
   (b) Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a twice continuously differentiable function so that $f''(t) \geq 0$ for every $t \in I$. Prove that f is convex on $[a, b]$; that is
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$
   whenever $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

5. (a) Suppose that A is a nonempty set of real numbers. Prove that the function $\Phi : \mathbb{R} \to \mathbb{R}$ given by
$$\Phi(x) := \inf\{|x - a| : a \in A\}$$
   is uniformly continuous on all of $\mathbb{R}$.
   (b) Suppose that $S \subset \mathbb{R}$ does not contain any open intervals. Suppose also that S contains all its limit points, i.e. if $x \in \mathbb{R}$ and there exists a sequence $x_n \in S$ whose limit is x, then $x \in S$. Prove that there exists a strictly increasing differentiable function f on all of $\mathbb{R}$ such that $f'(x) = 0$ if and only if $x \in S$. **Hint**: Use the function from part (a).

6. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of analytic functions on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and there exists $M > 0$ such that $\|f_n(z)\| \leq M$ for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} f_n(z)$ exists for all $z \in \mathbb{D}$ and let $f(z) := \lim_{n\to\infty} f_n(z)$. Prove that for any $\alpha < 1$, f_n converges uniformly to f on $D_\alpha := \{z \in \mathbb{C} : |z| \leq \alpha\}$.