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Each problem is worth a maximum of 2 points. The passing minimum score is 20.

1. Find the Moment Generating Function of an Gamma Random Variable with density

$$f(x|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x),\ x > 0.,\ \beta > 0.$$

and use it to get the variance.

2. Let X and Y be Uniform in the Unit Interval $(0,1)$ and independent. Find the density of $W = XY$.

3. Let $Y_{(1)} = \min(X_1, X_2, \ldots X_n)$ and $Y_{(n)} = \max(X_1, X_2, \ldots, X_n)$, where $X_1, X_2, \ldots, X_n$ are independent Exponential (β). Find the distribution and density of $Y_{(1)}$ and $Y_{(n)}$.

4. Let (X, Y) a two dimensional random variable. Prove that

$$Var[X] = Var[E[X|Y]] + E[Var[X|Y]].$$

5. Let X_1, X_2, X_3 random variables for which: $Var(X_i) = 10, i = 1,2,3$ and $Cov(X_i, X_j) = -1.$, for $i \neq j$. Find $Var(X_1 + X_2 - X_3)$.

6. Prove the Law of large numbers: suppose $X_1, \ldots, X_n$ are independent and identically distributed with mean μ and variance σ^2, and let $\bar{X}_n$ denote the sample mean. Then

$$\lim_{n\to\infty} P(|\bar{X}_n - \mu| < \varepsilon) = 1$$

Hint: Use Chebyshev inequality

7. Suppose that X_1, X_2 are independent exponentially distributed random variables, both with parameter β, and let $Y = X_1 + X_2$. Find the density of Y.

8. A disease affects a 0.01 of the population. There is a clinical test so that the probability of a positive test given that the person has the disease is 0.95 and the probability of a negative test given the person does not have the disease is 0.90. Suppose that the person gets positive in a test. What is the probability that the person is sick?

9. A machine has two parts that could fail and have to be replaced. The probability of failure of parts A and B are .05 and .10 respectively. If failures of these parts are independent of each other, what is the probability that at least one of them will fail?

10. Prove that the sum of ten independent Normal random variables is Normal.

11. When a new machine part is selected for installation, it is first inspected. The probability that the part fails the inspection (and consequently is not used) is equal to .05. If the part passes inspection and is used, its lifetime (i.e. the time until it becomes defective) is exponential with a mean of 100 hours. Before an inspection has taken place, what is the probability that the part will last at least 100 hours?

12. The number of claims that an insurance company gets follows a Poisson r.v. with $\lambda = 100$. The amounts of each claim is Uniformly distributed over $[0, 50]$. What is the probability that the claims are bigger that \$1000.? (You may use the Normal approximation).

13. A single fair die is rolled repeatedly. Let X be the random variable for the number of non-ones rolled before the fourth one is rolled. What

are $E(X)$ and $Var(X)$?

14. You have a Poisson variable with mean 5. What is the probability that the variable is exactly equal to 4 given that it is bigger than 0(zero)?

15. S is a Binomial variable with $n = 1000$ and $p = 0.20$. Approximate the probability that S is bigger that 250.