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The best 5 exercises will be marked.

1. The random variable Y is Student-t with d degrees of freedom, $Y \sim f_S(y|d)$.

   a) Which are the Mean and Variance of Y? When do they exist?

   b) Which is the distribution of Y^2?

   c) Show that: $\lim_{p \to \infty} f_S(y|p) \to Normal(y|0,1)$, at each y in the real line.

2. The sequence $y_1, y_2, \ldots$ is a vector of random variables that converges in probability to a constant a. Assume that $P(Y_i > 0) = 1$, for all i.

   a) Prove that the sequences $W_i = \sqrt{Y_i}$ and $T_i = \frac{c}{Y_i}$ converge in probability.

   b) Does $\frac{\sigma}{\sqrt{\sum y^2/n}}$ converge in probability to 1?

3. Let $Y_i \sim \text{Uniform}(0, \theta)$, $i = 1, \ldots, n$.

   a) Estimate θ by both the methods of Moments and Maximum Likelihood.

   b) Calculate the means and variances of both estimators.

   c) Assume a Uniform prior on the positive line for θ. For Quadratic Loss, which is the optimal Bayes Estimator?

4. Assume the Linear Regression through the origin: $Y_i = \alpha x_i + \varepsilon_i$, $i = 1, \ldots, n$, where ε_i are iid $N(0, \sigma^2)$ where σ^2 is unknown.

   a) Expose a sufficient statistics of dimension 2, for $[\alpha, \sigma^2]$.

b) Find a Maximum Likelihood estimator for α and show that it is an unbiased estimator for α.

c) Find the distribution of the Max. Likelihood Estimator of α.

d) Assume that the prior is $p(\alpha, \sigma) = 1/\sigma$. Which is the Bayesian optimal estimator for quadratic loss for α?

5. We have n Bernoulli θ random variables $Y_1, \ldots, Y_n$.

   a) Does the variance of $\bar{Y}$ attain the Cramer-Rao Lower Bound? Is $\bar{Y}$ the best unbiased estimator for α?

   b) Assume an Uniform prior for θ. For Quadratic Loss, which is the optimal Bayes estimator for θ?

6. Assume $Y_1, \ldots, Y_n$ is a random sample from a $N(\theta, \sigma^2)$, where θ, σ^2 are unknown. For each of the following hypothesis find the rejection region for an optimal test of Type 1 Error $\alpha = 0.05$.

   a) $H_0 : \theta = 0$; $H_1 : \theta \neq 0$;

   b) $H_0 : \theta \leq 0$ vs $H_1 : \theta > 0$

   c) In situation a) assume a prior like the following: under H_0, $\pi(\sigma) = 1/\sigma$ and under H_1, the prior is $p(\theta, \sigma) = p(\sigma) \times p(\theta|\sigma) = \frac{1}{\sigma} \times N(\theta|0, 2\sigma^2)$. Write an expression for the Bayes Factor.