Departamento de Matemáticas
Facultad de Ciencias Naturales
Recinto de Río Piedras

February 11, 2008

MS Qualifying Examination: Complex Analysis

Choose any three (3) of the following five (5) Problems.
Time for the Examination: 2.5 Hours

1. Let $f(z) := \dfrac{1}{z(z-1)(z-2)}$. Give the Laurent Expansion of $f(z)$ in each of the annuli:
   (a) (3 points) $\text{ann}(0,0,1) := \{z \in \mathbb{C} : \ 0 < |z| < 1\}$.
   (b) (3 points) $\text{ann}(0,1,2) := \{z \in \mathbb{C} : \ 1 < |z| < 2\}$.
   (c) (4 points) $\text{ann}(0,2,\infty) := \{z \in \mathbb{C} : \ 2 < |z| < \infty\}$.

2. (a) (5 points) Evaluate the following integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{4+x^2}\, dx$.
   (b) (5 points) Show that if f and g are analytic functions on a region G such that $\bar{f}g$ is analytic, then either f is constant or $g \equiv 0$.

3. Suppose that f and g are entire functions, and that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$.
   (a) (5 points) Show that if $g(z) \neq 0$ for all $z \in \mathbb{C}$, then there is a constant C such that $f(z) = Cg(z)$, for all $z \in \mathbb{C}$.
   (b) (5 points) Show that if $g(1/n) = 0$ for $n = 1, 2, \ldots$, then there is a constant C such that $f(z) = Cg(z)$, for all $z \in \mathbb{C}$.

4. (10 points) Suppose that f_n is a sequence of analytic functions on the unit disc $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$. Suppose also that the sequence is uniformly bounded, that is, there is an M such that
$$|f_n(z)| \leq M \quad \text{for all} \quad z \in \mathbb{D} \text{ and } \quad n \in \mathbb{N}.$$
   Suppose that $f(z)$ is a function such that
$$\lim_{n\to\infty} f_n(z) = f(z) \text{ for all } z \in \mathbb{D}.$$
   Prove that f_n converges to f uniformly on $\mathbb{D}_r := \{z \in \mathbb{C} : \ |z| \leq r\}$ for any $r < 1$.

5. (a) (5 points) Evaluate the following integral $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2}\, dx$.
   (b) (5 points) Let f be an analytic function on $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ and suppose that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Show that $|f'(0)| \leq 1$.