Universidad de Puerto Rico
Departamento de Matemáticas
Recinto de Río Piedras

Analisis Complejo **8 de Febrero 2006**

Qualifying Examination
Master Level

Choose any three of the following five Problems.
All the Problems have the same number of Points (10 Points).

Time for the Examination: Three (3) Hours

1. (a) (**5 points**). State and prove the Theorem of Goursat.

(b) (**5 points**). Consider the function $u:\ \mathbb{C} \to \mathbb{R}$ defined by

$$u(x+iy) := 1 + x + e^{-2y}\cos(2x), \qquad (x,y) \in \mathbb{R}^2.$$

(i) Prove that the function u is harmonic on $\mathbb{C}$.

(ii) Find a harmonic function $v:\ \mathbb{C} \to \mathbb{R}$ verifying the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

2. (a) (**5 points**). Show that any holomorphic function $f:\ \Omega \to \mathbb{C}$, defined on an open set $\Omega \subset \mathbb{C}$, can be expanded as a power series

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

which converges absolutely in the greatest open disc $B(a,r)$ centered at $a \in \Omega$ and included in Ω.

(b) (**5 points**). Let $\Omega \subset \mathbb{C}$ be an open set and $f:\ \Omega \to \mathbb{C}$ be a complex valued function with real and imaginary parts $u:\ \Omega \to \mathbb{R}$ and $v:\ \Omega \to \mathbb{R}$ (that is $f(z) = u(x,y) + iv(x,y),\ z = x+iy \in \mathbb{C}$). Show that f is complex differentiable at $z_0 = (x_0, y_0)$ if and only if u and v are real differentiable at z_0 and

$$\frac{\partial u(x_0,y_0)}{\partial x} = \frac{\partial v(x_0,y_0)}{\partial y} \quad \text{and} \quad \frac{\partial u(x_0,y_0)}{\partial y} = -\frac{\partial v(x_0,y_0)}{\partial x}.$$

3. (a) (**5 points**). Let

$$I(r) := \int_\gamma \frac{e^{iz}}{z}\, dz,$$

where $\gamma:\ [0,\pi] \to \mathbb{C}$ is defined by $\gamma(t) := re^{it}$. Show that

$$\lim_{r\to\infty} \int_\gamma \frac{e^{iz}}{z}\, dz = 0.$$

(b) (**2 points**). Let $\Omega := \{z \in \mathbb{C} : |z| < 4\}$ and let h be an analytic function on Ω such that $|h(z)| \leq 1$ for all $z \in \Omega$ with $|z| = 3$. Prove that

$$|h''(z)| \leq \frac{3}{4} \quad \text{for all } z \in \Omega \quad \text{with } |z| \leq 1.$$

(c) (**3 points**). Consider the function $g : \mathbb{C} \setminus \{-1, 2\} \to \mathbb{C}$ defined by $g(z) := \frac{3}{(z+1)(z-2)}$. Give the Laurent expansion of g at the point $z_0 = 1$ which converges at the point $z = i$.

4. (a) (**5 points**). Show that for $a > 1$,

$$\int_0^\pi \frac{1}{a + \cos(\theta)}\, d\theta = \frac{\pi}{\sqrt{a^2 - 1}}.$$

(b) (**5 points**). Consider the function $f : \mathbb{C} \setminus \{-1, 1\} \to \mathbb{C}$ defined by $f(z) := \frac{2\exp(z)}{z^2 - 1}$.

(i) Find all singularities z_0 of f and determine its nature.

(ii) Calculate the Residues of f at each point z_0 of (i).

(iii) Let $\gamma : [0, 1] \to \mathbb{C}$ be the path defined by $\gamma(t) := -1 + 3\exp(2\pi i t)$. Calculate $\int_\gamma f(z)\, dz$.

5. (a) (**5 points**). Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on the open subset $\Omega \subset \mathbb{C}$. For any $a \in \Omega$ define the greatest star sharped subset $\Omega_a \subset \Omega$. Show that it is open and show that f has a primitive (antiderivative) on Ω_a.

(b) (**5 points**). Let f be an entire function and let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial of grad n. Prove that:

(i) If

$$|f(z)| \leq C \cdot |p(z)|, \quad z \in \mathbb{C}, \tag{1}$$

for some constant $C > 0$, then there exists a constant $\lambda \in \mathbb{C}$ such that $f = \lambda p$.

(ii) If there exist two constants $R > 0$ and $C > 0$ such that (1) is satisfied for all $z \in \mathbb{C}$ with $|z| \geq R$, then f is a polynomial of grad $\leq n$.