Universidad de Puerto Rico
Departamento de Matemáticas

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MS Qualifying Exam- Complex Variables

Solve any three of the following five problems

(1) (a) Let U be an open subset of $\mathbb{C}$, and $a \in U$ be an isolated singularity for a holomorphic function $f : U \longrightarrow \mathbb{C}$. Prove that if a is not a pole of f, then $\overline{f(U \setminus \{a\})} = \mathbb{C}$.

(b) Evaluate the integral $\displaystyle\int_\gamma \frac{z^2+1}{z(z^2+4)}dz$ where $\gamma(t) = re^{it}$, $0 \leq t \leq 2\pi$, for all values of $r > 0$, $r \neq 2$.

(2) (a) State an prove the theorem of Goursat.

(b) Prove that $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$ where $a > 0$, by the *method of residues.*

(3) (a) Suppose γ_0 and γ_1 are closed rectifiable curves (with parameter interval $[0, 1]$) in the complex plane and there exists $\alpha \in \mathbb{C}$ such that for all $t \in [0, 1]$, $|\gamma_0(t) - \gamma_1(t)| < |\alpha - \gamma_0(t)|$. Prove that $n(\gamma_0, \alpha) = n(\gamma_1, \alpha)$ where $n(\gamma, a)$ is the index of the curve γ with respect to the point a.

Hint. Consider the path $\gamma := (\gamma_1 - \alpha)/(\gamma_0 - \alpha)$ and first verify in which component of $\mathbb{C} \setminus \gamma$ the point 0 is located.

(b) Let $a \in \mathbb{R} \setminus \{0\}$. Compute the integral $\displaystyle\int_0^\infty \frac{x^2 dx}{(x^2+a^2)^3}dx$ by the *method of residues.*

(4) (a) Show that the sum of a power series is holomorphic in the interior of its circle of convergence.

(b) Compute $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha\cos\theta + \alpha^2}$ where $|\alpha| \neq 1$.

(5) (a) Let f be an entire function such that $\displaystyle\lim_{|z|\to\infty} |f(z)| = \infty$. Prove that f is a polynomial.

(b) Compute the integral $\displaystyle\int_0^{\pi/2} \frac{dt}{a^2\cos^2 t + b^2\sin^2 t}$ by considering the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where $a, b > 0$.