April 30, 2003

Qualifying Examination
Complex Variables

Solve any three of the following five problems

1. Let (c_n) be a sequence of complex numbers.

   (a) Give an example where the series $\sum_{n=0}^{\infty} c_n$ diverges but $\lim_{n\to\infty} nc_n = 0$.

   From now on, we assume that the power series $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence $R = 1$ and we set $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for $|z| < 1$. We further assume that $\lim_{z\to 1} f(z) = s$ and $\lim_{n\to\infty} nc_n = 0$.

   (b) Let $|z| < 1$ and set $n = [\frac{1}{1-|z|}]$ (where $[x]$ is the greatest integer less than or equal to x). Let $S_1 = \sum_{k=n+1}^{\infty} c_k z^k$, $S_2 = \sum_{k=0}^{n} c_k(1-z^k)$. Verify that $S_1 - S_2 = f(z) - \sum_{k=0}^{n} c_k$ and that if $\varepsilon > 0$ is given, then for n large enough, $|S_1| < \varepsilon$.

   (c) Show that if (b_n) is a sequence of complex numbers with $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} \frac{b_0 + b_1 + ... + b_n}{n+1} = 0$.

   (d) Verify that for $|z| < 1$, $|1 - z^n| \leq n|1 - z|$.

   (e) Let $A > 0$ and suppose that $\frac{|1-z|}{1-|z|} \leq A$. Show that $|S_2| \leq \frac{A}{n} \sum_{k=0}^{n} |kc_k|$ and conclude that $\lim_{n\to\infty} \sum_{k=0}^{n} c_k = s$.

2. We let $D = \{z \in \mathbb{C}, |z| < 1\}$ be the unit disk in $\mathbb{C}$. Suppose f is a function holomorphic in D such that $f(0) = 0$ and $|f(z)| \leq 1$ in D.

   (a) i. Prove that $|f(z)| \leq |z|$ in D.

   ii. Suppose there exists $z_0 \in D$ such that $|f(z_0)| = 1$. Prove that there exists $\alpha \in \mathbb{R}$ such that $f(z) = e^{i\alpha}z$ for all $z \in D$.

   (b) Compute $\displaystyle\int_0^\infty \frac{x^2}{1+x^4}dx$.

3. (a) Suppose f is continuous on a simply connected domain $D \subset \mathbb{C}$ and $\displaystyle\int_\gamma f(z)dz = 0$ for each triangle Δ with boundary γ such that Δ and its interior are in D. Prove that f is holomorphic in D.

   (b) Suppose f is holomorphic on $D = \{z \in \mathbb{C}, |z| < 1\}$ and $|f(z)| \leq 1$ on D. Prove that $|f'(0)| \leq 1$.

   (c) Compute the integral $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{x^2+x+1}dx$.

4. (a) We let $D^* = \{z \in \mathbb{C}, 0 < |z| < 1\}$. Suppose f is holomorphic in D^* and for every circle $\gamma_r : \{|z| = r\}$ with $0 < r < 1$, we have $\int_\gamma f(z)dz = 0$. Does it follow that f has a holomorphic extension to $D = D^* \cup \{0\}$?

   (b) Compute that integral $\displaystyle\int_0^\infty \frac{\cos ax}{1+x^2}dx$ where $a \in \mathbb{R}$ is given.

   (c) Prove that the function f defined by $f(z) = \displaystyle\sum_{n=1}^\infty e^{-n}\sin\sqrt{n}z$ an entire function.

5. (a) Let γ be a closed rectifiable curve in $\mathbb{C}$. Define $n(\gamma;\, a) = \dfrac{1}{2\pi}\displaystyle\int_\gamma (z-a)^{-1}dz$ for a not on γ. Prove that on each (connected) component of $\Omega := \mathbb{C}\setminus\gamma$, the function $n(\gamma;\, .)$ is constant.

   (b) Compute $\displaystyle\int_0^\infty \frac{1}{(a^2+x^2)^2}dx$ where $a > 0$.