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MODERN ALGEBRA QUALIFYING EXAM

1. This test is divided in two (2) parts: *Group Theory* and *Ring Theory*. Each part consists of four (4) problems. The test has a total of eight (8) problems; each worth 10 points. The three best solutions of each part will be chosen.

2. Turn off the cell phone and any other electronic device.

3. Show your work. To get credit, your answers must be well-written, well-organized, and properly justified.

4. **MS students:** To pass, you should get at least ten (10) points in each part AND thirty (30) points overall.

5. **PHD students:** To pass, you should get at least fifteen (15) points in each part AND forty (40) points overall.

6. This is a 3 hours exam.

PART I: Group Theory

1. Prove that every group of order $20449 = 11^2 \cdot 13^2$ is an abelian group.

   Hint: Use Sylow Theorems.

2. If φ is an automorphisim of $\mathbb{Z}_n$, show that φ is multiplication by m for some m relatively prime to n. Conclude that $\text{Aut}(\mathbb{Z}_n) \simeq U_n$ (U_n is the group of units modulo n).

3. Let $Z(G)$ be the center of a group G. Show that if $G/Z(G)$ is a cyclic group, then G is abelian.

4. Let P be a p-group acting on a finite non-empty set X. Let

$$X^P = \{x \in X \mid g \cdot x = x \text{ for all } g \in P\}.$$

   Prove that $|X^P| \equiv |X| \mod p$.

PART II: Ring Theory

5. For an integral domain R, let $R[x]$ be the polynomial ring over R in the variable x. For $r \in R$, let $\phi_r : R[x] \to R$ be the evaluation homomorphism defined by $\phi_r(f(x)) = f(r)$

(a) Prove that if I is a proper ideal of $\mathbb{C}[x]$, then there exists $r \in \mathbb{C}$ with $\phi_r(I) \neq \mathbb{C}$.

(b) Let I be the ideal of $\mathbb{Z}[x]$ generated by 3 and $x^2 + 1$. Prove that I is a proper ideal of $\mathbb{Z}[x]$, and that $\phi_r(I) = \mathbb{Z}$ for every $r \in \mathbb{Z}$.

6. Find the degree of the splitting field E of $x^6 - 3$ over the following fields:

   (a) $\mathbb{Q}(\sqrt{3})$

   (b) $\mathbb{F}_7$, the field with 7 elements.

   (c) $\mathbb{F}_5$, the field with 5 elements.

7. Let R be a ring with unit 1. Suppose that the order of R is $|R| = p^2$ for some prime number p. Then prove that R is a commutative ring.

   Hint: Consider the set $Z = \{z \in R \,|\, zr = rz \text{ for any } r \in R\}$. Show that Z is a subgroup of the additive group structure of R.

8. Let $\mathbb{Z}[x]$ be the ring of polynomials in x with integer coeffiicients and $(x^2 + 1, x - 2)$ the ideal generated by the indicated polynomials. Prove that

$$\mathbb{Z}[x]/(x^2 + 1, x - 2)$$

   is the field with five elements.