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PH.D ALGEBRA QUALIFYING EXAM

1. This test is divided in two (2) parts: *Group Theory* and *Ring Theory*. Each part consists of four (4) problems. The test has a total of eight (8) problems.
2. Turn off the cell phone and any other electronic decive.
3. Show your work. To get credit, your answers must be well-written, well-organized, and properly justified.
4. **PHD students:** to pass, you should get at least fifteen (15) points in each part AND forty (40) points overall.
5. **MS students:** to pass, you should get at least ten (10) points in each part AND thirty (30) points overall.
6. This is a three (3) hour test.

PART I: Group Theory

1. Let A be a finite abelian group. Let n be the order of A and let m be the exponent of A (i.e., m is the least positive integer such that $a^m = e$ for all $a \in A$).
   (a) Show that $m|n$ (i.e., m is a factor of n). **(4 pts)**
   (b) Show that A is cyclic if and only if $m = n$. **(6 pts)**
2. Let G be a group of order 351. Then G has a normal Sylow p-subgroup for some prime p dividing 351. **(10 pts)**
3. A group G has exactly three subgroups if and only if it is cyclic of order p^2 for some prime p. (**10 pts**)
4. Do the following:
   (a) Prove or disprove: If $H \lhd G$ is a normal subgroup of G and G/H is cyclic, then G is an abelian group. (**5 pts**)
   (b) Prove that A_4 cannot have a normal subgroup of order 6. (**5 pts**)

PART II: Ring Theory

1. Consider the ring $R = \mathbb{Z}[x]$. Do the following:
   (a) Show that R is NOT a principal ideal domain (PID). **(6 pts)**
   (b) Show that if I is a maximal ideal of R, then I contains a $p \in \mathbb{Z}$ such that p is prime. **(4 pts)**
2. Do the follwing
   (a) Determine whether $x^4 + 1$ is irreducible or reducible over $\mathbb{F}_5$, where $\mathbb{F}_5$ is the finite field of 5 elements. (**5 pts**)

(b) Prove that the finite fields $\mathbb{F}_5[x]/I$ and $\mathbb{F}_5[x]/J$ are isomorphic, where $I = (x^3+3x^2+x+2)$ and $J = (x^3 + x + 1)$ are maximal ideals. **(5 pts)**

3. In a ring R we have $x^3 = x$ for all $x \in R$, then show that R is commutative.

4. Let $\mathbb{Z}_n$ be the ring of integers modulo n. Do the following:

   (a) Find explicitly all the ring homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}_{10}$. **(5 pts)**

   (b) Prove that $\mathbb{Z}_m$ is a ring of principal ideals. **(5 pts)**