University of Puerto Rico, Río Piedras
College of Natural Sciences
Department of Mathematics
San Juan, Puerto Rico

PH.D ALGEBRA QUALIFYING EXAM

1. This test is divided in two (2) parts: *Group Theory* and *Ring Theory*. Each part consists of four (4) problems. The test has a total of eight (8) problems.
2. Turn off the cell phone and any other electronic decive.
3. Show your work. To get credit, your answers must be well-written, well-organized, and properly justified.
4. To pass, you should get at least fifteen (15) points in each part AND forty (40) points overall.
5. This is a three (3) hour test.

PART I: Group Theory

1. Let V be an n-dimensional over a field F with $|F| = q = p^r$. Write $GL(n, q)$ to denote $GL(V)$, and $SL(n, q)$ for the special linear group. Let $M(n, q)$ the set of triangular matrices with 1's along the diagonal. Show that $M(n, q)$ is a Sylow p-subgroups of $GL(n, q)$. Find the number of sylow p-subgroups of $GL(n, q)$.
2.
   - Let G be a group of order 55. Prove that G is not simple.
   - Let G be a group of order $495 = 3^2 \cdot 5 \cdot 11$. Prove that G is not simple.
3. Suppose that G is a group. Let G act on a set X. Let Stab(x) be the stabilizer of x under the group action. Show that stabilizers of elements in the orbit of x are conjugate subgroups
4. Determine the center of D_{2n}. Show that a group can not be written as a union of two proper subgroups.

PART II: Ring Theory

1. Prove that $f(x) = x^4 + x^3 + x^2 + 6x + 1$ is irreducible over Q . Determine whether $f(x)$ is irreducible over the finite field with $2, 3, 5$ elements.
2. Let $\mathbb{F}_2$ be the field with two elements. Set
$$\begin{aligned} d(x) &= x^8 + x + 1 \\ f(x) &= x^{10} + x^9 + x^8 + x^3 + 1 \\ g(x) &= x^{11} + x^9 + x^8 + x^4 + x^3 + x^2 + 1 \end{aligned}$$
   (a) Show that $d(x) = GCD(f(x), g(x))$.
   (b) Show that $d(x)$ is not divisible by the square of a nonconstant polynomial.
3. Let R be a principal ideal domain and let I and J be nonzero ideals in R. Show that the product ideal $IJ = I \cap J$ if and only if they are comaximal i.e, $I + J = R$. Give example of a ring R and two comaximal ideals.
4. Let $\in F := GF(2)[x]/ < p(x) >$, where $p(x) := x^4 + x^3 + 1$. Express α^4 as a polynomial in α. Determine $|F|$, and order of α. Determine an i so that $\gamma := \alpha^i$ is a root of $q(x) := x^4 + x + 1$. What is the order of γ? Determine $r(x) \in GF(2)[x]$ so that α^{-1} is a root of $r(x)$.