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Hurewicz theorem

In mathematics, the **Hurewicz theorem** is a basic result of algebraic topology, connecting homotopy theory with homology theory via a map known as the **Hurewicz homomorphism**. The theorem is named after Witold Hurewicz, and generalizes earlier results of Henri Poincaré.

Contents

Statement of the theorems

The Hurewicz theorems are a key link between homotopy groups and homology groups.

Absolute version

For any path-connected space *X* and positive integer *n* there exists a group homomorphism

$$h_*\colon \pi_n(X) \to H_n(X),$$

called the **Hurewicz homomorphism**, from the *n*-th homotopy group to the *n*-th homology group (with integer coefficients). It is given in the following way: choose a canonical generator $u_n \in H_n(S^n)$, then a homotopy class of maps $f \in \pi_n(X)$ is taken to $f_*(u_n) \in H_n(X)$.

The Hurewicz theorem states cases in which the Hurewitz homomorphism is an isomorphism.

- For $n \geq 2$, if *X* is $(n-1)$-connected (that is: $\pi_i(X) = 0$ for all *i*<*n*), then $\tilde{H}_i(X) = 0$ for all *i*<*n*, and the Hurewicz map $h_*\colon \pi_n(X) \to H_n(X)$ is an isomorphism.[1]: 366, Thm.4.32 This implies, in particular, that the homological connectivity equals the homotopical connectivity when the latter is at least 1. In addition, the Hurewicz map $h_*\colon \pi_{n+1}(X) \to H_{n+1}(X)$ is an epimorphism in this case.[1]: 390, ?
- For $n = 1$, the Hurewicz homomorphism induces an isomorphism $\tilde{h}_*\colon \pi_1(X)/[\pi_1(X), \pi_1(X)] \to H_1(X)$, between the abelianization of the first homotopy group (the fundamental group) and the first homology group.

Relative version

For any pair of spaces (X, A) and integer $k > 1$ there exists a homomorphism

$$h_* \colon \pi_k(X, A) \to H_k(X, A)$$

from relative homotopy groups to relative homology groups. The Relative Hurewicz Theorem states that if both X and A are connected and the pair is $(n-1)$-connected then $H_k(X, A) = 0$ for $k < n$ and $H_n(X, A)$ is obtained from $\pi_n(X, A)$ by factoring out the action of $\pi_1(A)$. This is proved in, for example, Whitehead (1978) by induction, proving in turn the absolute version and the Homotopy Addition Lemma.

This relative Hurewicz theorem is reformulated by Brown & Higgins (1981) as a statement about the morphism

$$\pi_n(X, A) \to \pi_n(X \cup CA),$$

where CA denotes the cone of A. This statement is a special case of a homotopical excision theorem, involving induced modules for $n > 2$ (crossed modules if $n = 2$), which itself is deduced from a higher homotopy van Kampen theorem for relative homotopy groups, whose proof requires development of techniques of a cubical higher homotopy groupoid of a filtered space.

Triadic version

For any triad of spaces $(X; A, B)$ (i.e., a space *X* and subspaces *A*, *B*) and integer $k > 2$ there exists a homomorphism

$$h_* \colon \pi_k(X; A, B) \to H_k(X; A, B)$$

from triad homotopy groups to triad homology groups. Note that

$$H_k(X; A, B) \cong H_k(X \cup (C(A \cup B))).$$

The Triadic Hurewicz Theorem states that if *X*, *A*, *B*, and $C = A \cap B$ are connected, the pairs (A, C) and (B, C) are $(p-1)$-connected and $(q-1)$-connected, respectively, and the triad $(X; A, B)$ is $(p+q-2)$-connected, then $H_k(X; A, B) = 0$ for $k < p+q-2$ and $H_{p+q-1}(X; A)$ is obtained from $\pi_{p+q-1}(X; A, B)$ by factoring out the action of $\pi_1(A \cap B)$ and the generalised Whitehead products. The proof of this theorem uses a higher homotopy van Kampen type theorem for triadic homotopy groups, which requires a notion of the fundamental $\mathbf{cat}^n$-group of an *n*-cube of spaces.

Simplicial set version

The Hurewicz theorem for topological spaces can also be stated for *n*-connected simplicial sets satisfying the Kan condition.[2]

Rational Hurewicz theorem

Rational Hurewicz theorem:[3][4] Let *X* be a simply connected topological space with $\pi_i(X) \otimes \mathbb{Q} = 0$ for $i \leq r$. Then the Hurewicz map

$$h \otimes \mathbb{Q}\colon \pi_i(X) \otimes \mathbb{Q} \longrightarrow H_i(X;\mathbb{Q})$$

induces an isomorphism for $1 \leq i \leq 2r$ and a surjection for $i = 2r+1$.

Notes

1. Hatcher, Allen (2001), *Algebraic Topology*, Cambridge University Press, ISBN 978-0-521-79160-1
2. Goerss, Paul G.; Jardine, John Frederick (1999), *Simplicial Homotopy Theory*, Progress in Mathematics, vol. 174, Basel, Boston, Berlin: Birkhäuser, ISBN 978-3-7643-6064-1, III.3.6, 3.7
3. Klaus, Stephan; Kreck, Matthias (2004), "A quick proof of the rational Hurewicz theorem and a computation of the rational homotopy groups of spheres", *Mathematical Proceedings of the Cambridge Philosophical Society*, **136** (3): 617–623, doi:10.1017/s0305004103007114 (https://doi.org/10.1017%2Fs0305004103007114)
4. Cartan, Henri; Serre, Jean-Pierre (1952), "Espaces fibrés et groupes d'homotopie, II, Applications", *Comptes rendus de l'Académie des Sciences*, **2** (34): 393–395

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