Helton-Howe Trace, Connes-Chern Character, and Quantization

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Washington University in St. Louis

August 8, 2022

Outline

In this talk, we will report our recent study of the Helton-Howe trace and the Connes-Chern character for Toeplitz operators on weighted Bergman spaces. We will present a proof of the Helton-Howe trace and its generalizations via the idea of Toeplitz quantization.

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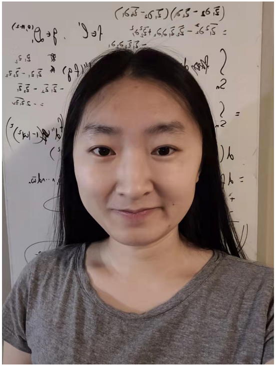
1. Toeplitz operators and the Helton-Howe trace formula
2. The Connes-Chern character
3. Toeplitz quantization and trace formulas

This talk is based on joint work with Yi Wang and Dechao Zheng.

My collaborators



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Proposition

The commutator

$$[T_f, T_g]$$

is a compact operator on $L^2_a(\mathbb{D})$.

Extension and K-homology

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In the Brown-Douglas-Fillmore theory, the above extension defines a K-homology class $[\mathcal{T}(\mathbb{D})]$ in $K_1(S^1)$.

Theorem

In $K_1(S^1)$, $[\mathcal{T}(\mathbb{D})] = [\frac{1}{i}\frac{d}{d\theta}]$.

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A direct calculation shows that the commutator $[T_z, T_z^*]$ is a trace class operator on $L_a^2(\mathbb{D})$. And this property extends to all $f, g \in C^\infty(\overline{\mathbb{D}})$.

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Theorem (Helton-Howe)

For $f, g \in C^\infty(\overline{\mathbb{D}})$,

$$\mathrm{tr}\left([T_f, T_g]\right) = \frac{1}{2\pi i}\int_{\mathbb{D}} \mathrm{d}f \wedge \mathrm{d}g.$$

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This result is deeply connected to the Pincus function for a pair of noncommuting selfadjoint operators.

Weighted Bergman space

Consider the probability measure $\mathrm{d}\lambda_t(z)$ $(t > -1)$ on $\mathbb{D}$:

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If we evaluate the trace on $K_1(C(S^1))$,

$$\mathrm{tr}([T_{e^{in\theta}}^{(t)}, T_{e^{-in\theta}}^{(t)}]) = -n.$$

The question is about the rigidity property at the level of cocycle/cochain instead of "cohomology".

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$$\mathrm{d}\lambda_t(z) = \frac{(n-1)!}{\pi^n B(n, t+1)}(1 - |z|^2)^t \mathrm{d}m(z),$$

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The full antisymmetrization $[T_{f_1}^{(0)}, ..., T_{f_{2n}}^{(0)}]$ is a trace class operator, and

$$\operatorname{tr}\left([T_{f_1}^{(0)}, ..., T_{f_{2n}}^{(0)}]\right) = \frac{n!}{(2\pi i)^n} \int_{\mathbb{B}_n} \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \cdots \wedge \mathrm{d}f_{2n}.$$

Cyclic cohomology and Connes-Chern character

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MR0823176, Connes, Alain, Noncommutative differential geometry, *Inst. Hautes Études Sci. Publ. Math.* No. **62** (1985), 257–360.

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of all (continuous) $(k+1)$-linear functionals on A.

Definition

Define the Hochschild codifferential $\partial \colon C^k(A) \to C^{k+1}(A)$ by

$$\begin{aligned}
&\partial\Phi(a_0 \otimes \cdots \otimes a_{k+1}) \\
&= \sum_{i=0}^{k} (-1)^i \Phi(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{k+1}) \\
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The Hochschild cohomology of A is the cohomology of the cochain complex $(C^\bullet(A), \partial)$.

Cyclic cohomology

Definition

A Hochschild cochain $\Phi \in C^k(A)$ is *cyclic* if for all $a_0, \ldots, a_k \in A$,

$$\Phi(a_k, a_0, \ldots, a_{k-1}) = (-1)^k \Phi(a_0, a_1, \ldots, a_k).$$

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Theorem (Connes-Hochschild-Kostant-Rosenberg)

$$HH^\bullet(C^\infty(M)) = \mathcal{D}^{deRham}_\bullet(M), \;\; HP^\bullet(C^\infty(M)) = H^{deRham}_\bullet(M).$$

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Cyclic cohomology pairs naturally with K-theory of the algebra.

The Connes-Chern character

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$$\tau_t(f_0, \cdots, f_{2p-1}) := \operatorname{tr}\left(\sigma_t(f_0, f_1) \cdots \sigma_t(f_{2p-2}, f_{2p-1})\right) - \operatorname{tr}\left(\sigma_t(f_1, f_2) \cdots \sigma_t(f_{2p-1}, f_0)\right).$$

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Up to a constant c, τ_t is the Connes-Chern character for the Schatten-p extension,

$$0 \longrightarrow \mathcal{S}_p \longrightarrow \mathcal{E} \to C^\infty(\partial\overline{\mathbb{B}_n}) \longrightarrow 0,$$

with $c = (-1)^{p-1}(2i\pi)^p(p - \frac{1}{2}) \cdots (\frac{3}{2})(\frac{1}{2})$.

Cyclic cocycle

Theorem (Connes)

The functional τ_t satisfies the following properties.

1) $\tau_t(f_1, \cdots, f_{2p-1}, f_0) = -\tau_t(f_0, f_1, \cdots, f_{2p-1})$

2) $\tau_t(f_0 f_1, f_2, \cdots, f_{2p}) - \tau_t(f_0, f_1 f_2, \cdots, f_{2p}) +$
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In general, Connes introduced cyclic cohomology as the receptacle of the Connes-Chern character of a Fredholm module.

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Remark

The Helton-Howe trace $\mathrm{tr}\left([T^{(0)}_{f_1}, ..., T^{(0)}_{f_{2n}}]\right)$ defines a cyclic cocycle on $C^\infty(S^{2n-1})$.

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1. Compute the explicit formula for the trace of the full antisymmetrization $[T^{(t)}_{f_1}, ..., T^{(t)}_{f_{2n}}]$. i.e.

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Does it depend on t?

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2. Recall

$$\tau_t(f_0, \cdots, f_{2p-1}) := \mathrm{tr}\left(\sigma_t(f_0, f_1) \cdots \sigma_t(f_{2p-2}, f_{2p-1})\right) - \mathrm{tr}\left(\sigma_t(f_1, f_2) \cdots \sigma_t(f_{2p-1}, f_0)\right).$$

   Compute the local expression of τ_t by taking the limit $t \to \infty$.

The semicommutator

Recall that in the Connes-Chern character, the key ingredient is the semicommutator $\sigma_t(f, g)$,

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$$T : C^\infty(\overline{\mathbb{B}_n}) \to B(L^2_{a,t}(\mathbb{B}_n)).$$

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A "quantization map" relating a classical mechanic system to its quantum version can be described by a linear map

$$Q^{\hbar} : C_c^{\infty}(M) \to B(\mathcal{H}_{\hbar}).$$

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A symplectic manifold M is the phase space of a classical mechanic system. The physical observables of this system are functions on the symplectic manifold.
A quantum mechanic system is described by a Hilbert space $\mathcal{H}$. The physical observables are represented by self-adjoint operators on the Hilbert space.
A "quantization map" relating a classical mechanic system to its quantum version can be described by a linear map

$$Q^{\hbar} : C_c^{\infty}(M) \to B(\mathcal{H}_{\hbar}).$$

The quantization $Q^{\hbar}$ is related to the original symplectic manifold via the following property.

$$||[Q_f^{\hbar}, Q_g^{\hbar}] - \hbar Q_{\{f,g\}}^{\hbar}||_{B(\mathcal{H}_{\hbar})} = o(\hbar).$$

Asymptotic expansion

In quantization, the following asymptotic expansion formula has been established.

$$||T_f^{(t)}T_g^{(t)} - \sum_{j=0}^{k} t^{-j} T^{(t)}_{C_j(f,g)}||_{B(L^2_{a,t})} = O(t^{-k-1}),\ t \to \infty,$$

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$$\begin{aligned} & C_1(f,g) \\ = & -i(1-|z|^2)\left[\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\frac{\partial g}{\partial \bar{z}_j} - (\sum_j \bar{z}_j \frac{\partial f}{\partial z_j})(\sum_{j'} z_{j'} \frac{\partial g}{\partial z_{j'}})\right] \end{aligned}$$

Expansion in Schatten-p norm I

For our study of the trace formula, we need to estimate Schatten-p norm of the asymptotic expansion formula.

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Theorem (T-Wang-Zheng)

Suppose $t > -1$, k is a non-negative integer and $\forall f, g \in \mathscr{C}^{k+1}(\overline{\mathbb{B}_n})$. Then we have the decomposition

$$T_f^{(t)} T_g^{(t)} = \sum_{l=0}^{k} c_{l,t} T_{C_l(f,g)}^{(t)} + R_{f,g,k+1}^{(t)}.$$

For any $t > -1$ and $k \geq 0$, the following hold.

(i) *If $n > 1$ then $R_{f,g,k+1}^{(t)} \in \mathcal{S}^p$ for any $p > n$.*

(ii) *If $n = 1$ then $R_{f,g,k+1}^{(t)} \in \mathcal{S}^1$.*

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(d) if $n = 1$, then for any $p \geq 1$,

$$\|R^{(t)}_{f,g,k+1}\|_{\mathcal{S}^p} \lesssim_{k,p} t^{-k-1+\frac{n}{p}}.$$

The case of unit disk

Let's start with the 1-dim case. Using the previous expansion formula, we can compute the trace of the semicommutator $\sigma_t(f,g) := T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}$ on $L^2_{a,t}(\mathbb{D})$.

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Theorem (T-Wang-Zheng)

$$\operatorname{tr}\left(T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}\right) = \frac{1}{2\pi i}\int_{\mathbb{D}} \partial f \wedge \bar{\partial} g + \int_{\mathbb{D}^2} \rho_t(|\varphi_z(w)|^2)\Delta f(z)\Delta g(w)\mathrm{d}m(z,w),$$

where ρ_t is a strictly positive function on $(0,1)$.

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Corollary

$$\operatorname{tr}[T_f^{(t)}, T_g^{(t)}] = \frac{1}{2\pi i}\int_{\mathbb{D}} \mathrm{d}f \wedge \mathrm{d}g.$$

Large t-limit (the disk case)

We take the limit of $t \to \infty$ in the following equation.

$$\begin{aligned}\operatorname{tr}\left(T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)}\right) &= \frac{1}{2\pi i}\int_{\mathbb{D}} \partial f \wedge \bar{\partial} g \\ &\quad + \int_{\mathbb{D}^2} \rho_t(|\varphi_z(w)|^2)\Delta f(z)\Delta g(w)\mathrm{d}m(z,w) \\ &\longrightarrow \frac{1}{2\pi i}\int_{\mathbb{D}} \partial f \wedge \bar{\partial} g,\ t \to \infty.\end{aligned}$$

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- The above formula suggests that in general the Connes-Chern character could depend on t.
- The above cochain is not a Hochschild cocycle, but contain interesting information about the holomorphic/complex structure.

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$$|\tau_t(f_0, \cdots, f_{2p-1})| \lesssim t^{-1} \longrightarrow 0,\ t \to \infty.$$

This estimate suggests that we consider the case of $p = n$. However,

$$\sigma_t(z_1, \bar{z}_1) \cdots \sigma_t(z_n, \bar{z}_n) - \sigma_t(\bar{z}_1, z_2)\sigma_t(\bar{z}_2, z_3) \cdots \sigma_t(\bar{z}_n, z_1)$$

is not a trace class operator.

Leading term

For $f, g \in \mathscr{C}^2(\mathbb{B}_n)$, define

$$C_1(f,g) := -i(1-|z|^2)\Big[\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\frac{\partial g}{\partial \bar{z}_j} - \Big(\sum_j \bar{z}_j \frac{\partial f}{\partial z_j}\Big)\Big(\sum_{j'} z_{j'} \frac{\partial g}{\partial z_{j'}}\Big)\Big].$$

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Theorem (T-Wang-Zheng)

When $t \to \infty$, the limit of $\operatorname{tr}\big(\sigma_t(f_1,g_1)\cdots\sigma_t(f_{n+1},g_{n+1})\big)$ *has the following leading term*

$$\begin{aligned} & t^{-1}\operatorname{tr}\left(T^{(t)}_{C_1(f_1,g_1)\cdots C_1(f_{n+1},g_{n+1})}\right) \\ \sim & \frac{i^n}{\pi^n}\int_{\mathbb{B}_n} \frac{C_1(f_1,g_1)\cdots C_1(f_{n+1},g_{n+1})(z)}{(1-|z|^2)^{n+1}} \mathrm{d}m(z). \end{aligned}$$

Partial antisymmetrization

For $f_1, \ldots, f_n, g_1, \ldots, g_n \in L^\infty(\overline{B}_n)$ and $t > -1$, define the following partial anti-symmetric sums.

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$$[T^{(t)}_{f_1}, T^{(t)}_{g_1}, \ldots, T^{(t)}_{f_n}, T^{(t)}_{g_n}]^{\mathrm{odd}} = \sum_{\tau \in S_n} \mathrm{sgn}(\tau)\sigma_t(f_{\tau(1)}, g_1) \ldots \sigma_t(f_{\tau(n)}, g_n),$$

$$[T^{(t)}_{f_1}, T^{(t)}_{g_1}, \ldots, T^{(t)}_{f_n}, T^{(t)}_{g_n}]^{\mathrm{even}} = \sum_{\tau \in S_n} \mathrm{sgn}(\tau)\sigma_t(f_1, g_{\tau(1)}) \ldots \sigma_t(f_n, g_{\tau(n)}).$$

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Theorem (T-Wang-Zheng)

Suppose $t \geq -1$ and $f_1, g_1, \ldots, f_n, g_n \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. Then both $[T_{f_1}^{(t)}, T_{g_1}^{(t)}, \ldots, T_{f_n}^{(t)}, T_{g_n}^{(t)}]^{odd}$ and $[T_{f_1}^{(t)}, T_{g_1}^{(t)}, \ldots, T_{f_n}^{(t)}, T_{g_n}^{(t)}]^{even}$ are in the trace class.

Large t limit and the Helton-Howe trace

Theorem (T-Wang-Zheng)

For $f_1, g_1, \cdots, f_n, g_n \in \mathscr{C}^2(\overline{\mathbb{B}_n})$,

$$\begin{aligned}&\lim_{t\to\infty} \mathrm{tr}([T_{f_1}^{(t)}, T_{g_1}^{(t)}, \ldots, T_{f_n}^{(t)}, T_{g_n}^{(t)}]^{odd})\\ =&\lim_{t\to\infty} \mathrm{tr}([T_{f_1}^{(t)}, T_{g_1}^{(t)}, \ldots, T_{f_n}^{(t)}, T_{g_n}^{(t)}]^{even})\\ =&\frac{1}{(2\pi i)^n}\int_{\mathbb{B}_n} \partial f_1 \wedge \overline{\partial} g_1 \wedge \ldots \wedge \partial f_n \wedge \overline{\partial} g_n.\end{aligned}$$

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Theorem (T-Wang-Zheng)

Suppose $f_1, f_2, \dots, f_{2n} \in \mathscr{C}^2(\overline{\mathbb{B}_n})$ *and* $t \geq -1$. *Then*

$$\operatorname{tr}[T^{(t)}_{f_1}, T^{(t)}_{f_2}, \dots, T^{(t)}_{f_{2n}}] = \frac{n!}{(2\pi i)^n}\int_{\mathbb{B}_n} \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \ldots \wedge \mathrm{d}f_{2n}.$$

Thank you for your attention!