Topics Covered in Math 3151 Exam II

1. The student must be able to state all definitions.

2. The student must know all statement(s) of every theorem with a name.

3. The student must know the definition of the derivative and its applications. For example,

   In exercises a)-f), for what values of *A, B* is *f* differentiable at the branch point.

   a) $f(x)=\begin{cases} Ax+B, & x<0 \\ \cos x, & x \geq 0 \end{cases}$

   b) $f(x)=\begin{cases} \sin x, & 0 \leq x \leq 2\pi/3 \\ Ax+B, & 2\pi/3 < x \leq 2\pi \end{cases}$

   c) $f(x)=\begin{cases} 1+A\cos x, & x \leq \pi/3 \\ B+\sin(x/2) & x > \pi/3 \end{cases}$

   d) $f(x)=\begin{cases} x^3, & x \leq 1 \\ Ax+B, & 1<x \end{cases}$

      i) Find A, B so that f is differentiable at 1.

      ii) Using the A, B found in i), show that $y = Ax + B$ is the tangent line to x^3 at $(1,1)$.

      iii) Sketch the graph of f, carefully labeling its two branches.

   e) $f(x)=\begin{cases} Ax^3+Bx+2, & x \leq 2 \\ Bx^2-A & x>2 \end{cases}$

   f) $f(x)=\begin{cases} Ax^2+B, & x<-1 \\ Bx^5+Ax+4, & x \geq -1 \end{cases}$

4. The student must be able to find tangent and normal lines.

   Def: Let $y = f(x)$ be defined for $|x-c|<p$, some $p>0$. The tangent line to $y=f(x)$ through $(c, f(c))$ is that

   - Non-vertical line through $(c, f(c))$ with slope $m_{\tan}$ that is defined by
     $$\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}, \text{ provided this limit exists}$$
   - Vertical line through $(c, f(c))$, which has no slope, provided f is continuous at c and
     $$\lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = \infty \quad \text{or} \quad \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = -\infty$$

5. The student must know the proof that Differentiability Implies Continuity.

6. The student must know the different notations for the derivative: $f'(x), \frac{dy}{dx}$ and $D_x f(x)$

7. The student must be able to use and recognize the use of the differentiation rules. He/she must also know their names. The names are:

   a) Constant Rule
   b) Identity Rule
   c) Constant Multiple Rule
   d) Sum Rule
   e) Difference Rule
   f) Product Rule
   g) Reciprocal Rule
   h) Quotient Rule
   i) Power Rule (including negative exponents)
   j) Radical Rule
   k) Chain Rule

   The Chain Rule: Let $y = f(t)$ and $t = g(x)$. If g is differentiable at x and f is differentiable at $t = g(x)$, then the composition $f \circ g$ is differentiable at x and

$$D_x(f \circ g)(x) = D_t f(t) \cdot D_x g(x)$$

or

$$D_x y = D_t y \cdot D_x t$$

or

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

   Example: Let $y = f(t) = (t^3 + t^2 + t + 1)^6$ and $t = \cos x + \sin x$. Then, from $D_x y = D_t y \cdot D_x t$,

   we obtain $D_x y = D_t (t^3 + t^2 + t + 1)^6 \cdot D_x(\cos x + \sin x) =$

$$6 \cdot (t^3 + t^2 + t + 1)^5 \cdot (3t^2 + 2t + 1)(\cos x - \sin x) = 6((\cos x + \sin x)^3 + (\cos x + \sin x)^2 + (\cos x + \sin x) + 1)^5 \cdot (3(\cos x + \sin x)^2 + 2(\cos x + \sin x) + 1)(\cos x - \sin x)$$

8. The student must be able to reproduce the proofs of the differentiation rules a)-f) in 7. above.

9. The student must know the derivatives of the trigonometric, exponential, and logarithmic functions, the latter to the base e only.

10. The student must know how to work problems involving derivatives of higher order.

    For example, determine the following derivatives

    a) $\frac{d}{dx}\left[x \frac{d^2}{dx^2}(x - x^4)\right] =$

b) $\frac{d^2}{dx^2}\left[\left(x^3 - 3x\right)\frac{d}{dx}\left(x - x^{-1}\right)\right] =$

c) $\frac{d^4}{dx^4}\left[\sin^5 x - \cos^5 x\right] =$

11. The student must be able to solve problems on speed, velocity and acceleration as applied to rectilinear motion. Depending on the sign of the velocity and acceleration, the student must be able to determine the characteristics of the motion. The student should intuitively understand the following relationships. A proof of the relationships c)-f) below will be given later in the course using the Mean Value Theorem. Let v, a denote the velocity and acceleration, respectively. Then the relationships are

- $v > 0 \Leftrightarrow$ moving in the positive direction.
- $v < 0 \Leftrightarrow$ moving in the negative direction.
- $v > 0, a > 0 \Leftrightarrow$ speeding up in the positive direction.
- $v > 0, a < 0 \Leftrightarrow$ slowing down in the positive direction.
- $v < 0, a < 0 \Leftrightarrow$ speeding up in the negative direction.
- $v < 0, a > 0 \Leftrightarrow$ slowing down in the negative direction.

Exercises:

a) An object moves along the $x-$axis, its position at each time t given by the equation of motion

$$x(t) = t^3 - 12t^2 + 36t - 27$$

i) When is the object speeding up in the positive direction?

ii) When is the object speeding up in the negative direction?

iii) When is the object slowing down in the positive direction?

iv) When is the object slowing down in the negative direction?

b) Find the position, velocity, acceleration and speed at time $t_0 = 3$ of $x(t) = \frac{2t}{2+3}$.

c) An object moves along the coordinate line and its position at each time $t \geq 0$ is given by $x(t) = t + \frac{5}{t+2}$. Determine when, if ever, the object changes direction.

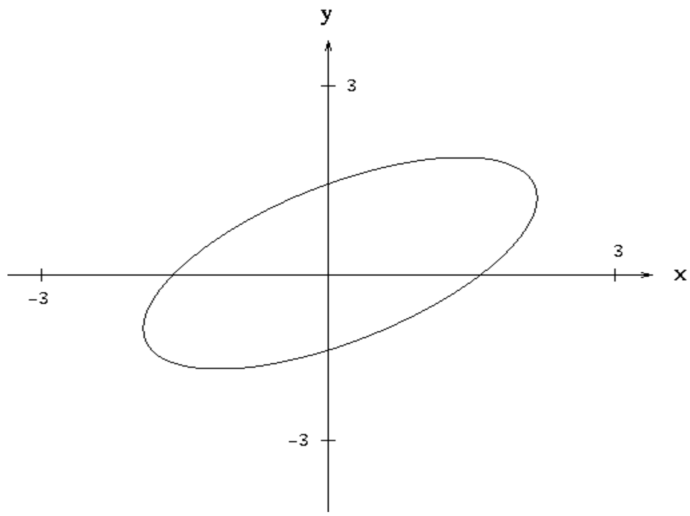
d) Galileo's formula for free fall near the surface of the earth is given by $y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$, where y_0 is the height of the object at $t = 0$, v_0 is the velocity of the object at $t = 0$, and $-g$ is the negative acceleration due to gravity ($g = 32$ feet per second per second $= 9.8$ meters per second per second).

i) A stone is dropped from a height of 98 meters. In how many seconds does it hit the ground? What is the speed at the instant of impact?

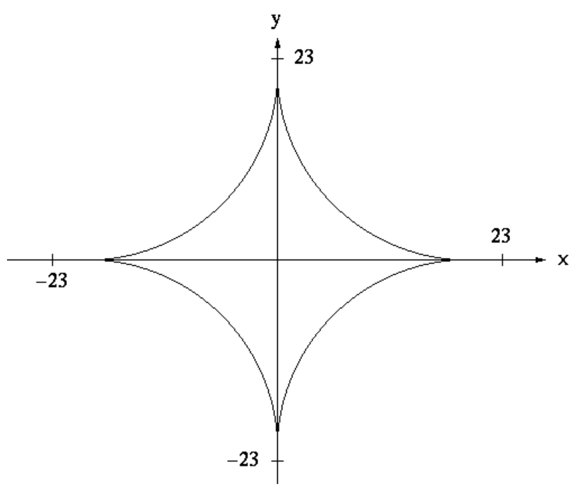
ii) An object is thrown upwards with an initial velocity of 72 feet per second.
   i. In how many seconds does it attain maximum height?
   ii. What is the maximum height?
   iii. What is the speed of the object as it reaches a height of 32 feet (1) going up? (2) coming back down?

12. The student must be able to differentiate implicitly and find tangents to curves at a point such as:

a) The graph of $x^2 - xy + y^2 = 3$ is a "tilted" ellipse (See diagram.). Among all points (x, y) on this graph, find the largest and smallest values of y. Among all points (x, y) on this graph, find the largest and smallest values of x.



b) Find all points (x, y) on the *astroid* $x^{2/3} + y^{2/3} = 8$ where lines tangent to the graph at (x, y) have slope -1.



c) Find the equation of the tangent line to the Devil's Curve $y^4 - 4y^2 = x^4 - 9x^2$ at the point $(2,3)$.

Devil's Curve