The Limit Concept

Notes by

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The concept of the limit is the most important concept in the Calculus. Everything that we will do depends upon it. This concept, historically, began with the work of Archimedes (287 BC – 212 BC) and was developed into the Calculus with the work of Sir Isaac Newton (1643-1727) and Gottfried Leibniz (1646-1716). Even though these two mathematicians were masters of intuition, their theory of the Calculus had many unjustifiable ideas. Finally, the Calculus was placed on a solid foundation with the work of such prominent mathematicians as Augustin Louis Cauchy (1789-1857), Bernard Bolzano (1781-1848), Karl Weierstrass (1815-1897) and Richard Dedekind (1831-1916). The Calculus that we will learn is actually due to Cauchy while the notation that we will use is due to Leibniz. One might say that it took over 150 years to logically elucidate the concept of the limit. So, if you find this concept somewhat difficult to understand in the beginning it is quite natural. But with persistence and motivation you will grasp it. A young physicist once complained to John von Neumann, a giant of a mathematician of the twentieth century, that he did not understand a certain mathematical method. "Young man", responded von Neumann, "in mathematics you don't understand things. You just get used to them".

We will approach our understanding of the limit concept from two perspectives. We will begin by presenting an intuitive discussion of the central theme of this concept. After this is absorbed, we will present the limit concept the way mathematicians understand and work with it. This latter presentation will assume that you have a solid understanding of certain concepts from the Pre-Calculus. Specifically, you should feel comfortable working with inequalities and the absolute value; in particular, with the triangle inequality.

Intuitive Discussion of the Limit

In general, we will be concerned with analyzing the behavior of a function f near a point c.

The Left Hand Limit

We assume that we have a function f that is defined near c^- (left side of c) but not necessarily at c; that is, $f(c)$ may not be defined. Let us imagine the following scenario. Let us imagine that we are the variable x walking on the x – axis approaching c from the left. There are three rules we must follow:

1. We begin approaching c from the left $\left(c^-\right)$ through x's where f is defined near c^-.
2. When we step on a number x, we observe the height $f(x)$.
3. We must never step on c itself.

We then ask the following question: As $x \neq c$ approaches c from the left $\left(x \to c^-\right)$, what number L, if any, do the heights $f(x)$ approach $\left(f(x) \to ?\right)$? If L exists, we write $\lim\limits_{x \to c^-} f(x) = L$, read "*The limit of* $f(x)$, *as* x *approaches* c *from the left, is* L". The number L is called the **left-hand limit** of f at c.

As an example refer to Figure 1, in which $c = 2$. Observe that $L = 5$ here.

The Left Hand Limit $\lim_{x \to 2^-} f(x) = 5$

$$f(x) = \begin{cases} x^2 + 1 & -2 \le x < 2 \\ 4 & x = 2 \\ \sin\left(\frac{\pi}{4}x\right) + 2 & 2 < x \le 6 \end{cases}$$

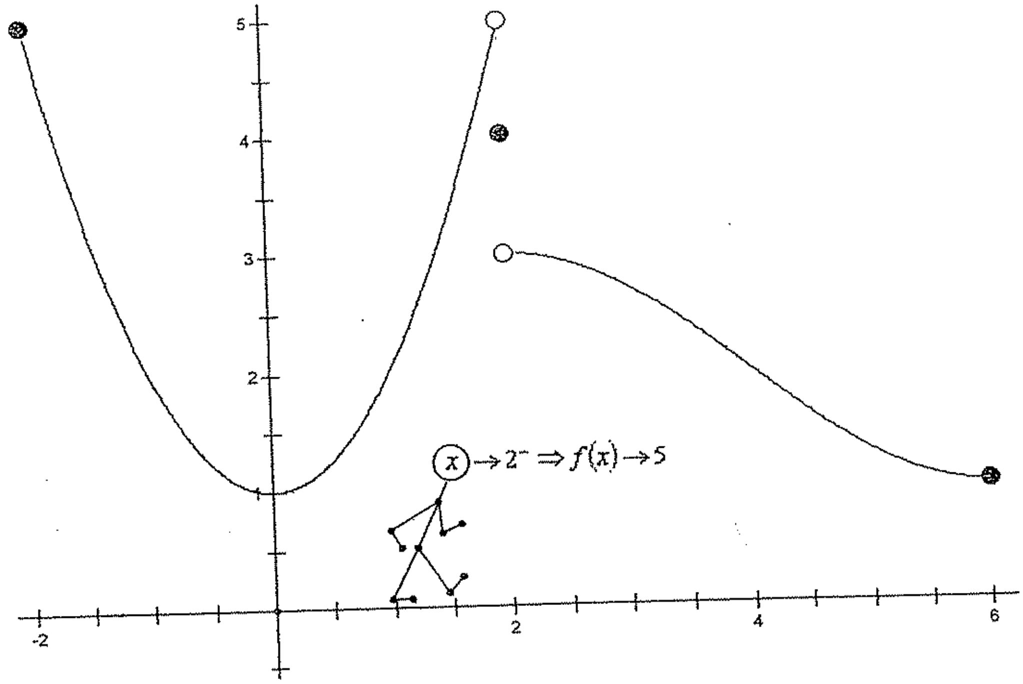

Figure 1

In an analogous manner, we have

The Right Hand Limit

We assume that we have a function f that is defined near c^+ (right side of c) but not necessarily at c; that is, $f(c)$ may not be defined. Let us imagine the following scenario. Let us imagine that we are the variable x walking on the $x-$axis approaching c from the right. There are three rules we must follow:

1. We begin approaching c from the right (c^+) through x's where f is defined near c^+.
2. When we step on a number x, we observe the height $f(x)$.
3. We must never step on c itself.

We then ask the following question: As $x \ne c$ approaches c from the right $(x \to c^+)$, what number R, if any, do the heights $f(x)$ approach $(f(x) \to ?)$? If R exists, we write $\lim_{x \to c^+} f(x) = R$, read "*The limit of* $f(x)$, *as* x *approaches* c *from the right, is* R". The number R is called the **right-hand limit** of f at c.

As an example refer to Figure 2, in which $c = 2$. Observe that $R = 3$ here.

Refer to Figure 3 to see a numerical example that corresponds to the graphical examples in Figure 1 and Figure 2. In Figure 3, the steps are numbered from 1 to ∞. If the handed limits exist, they must be unique, and it does not matter what the actual values of x are when $x \to c^-$ or $x \to c^+$.

The Right Hand Limit $\lim_{x \to 2^+} f(x) = 3$

$$f(x)=\begin{cases} x^2+1 & -2 \le x < 2 \\ 4 & x=2 \\ \sin\left(\frac{\pi}{4}x\right)+2 & 2 < x \le 6 \end{cases}$$

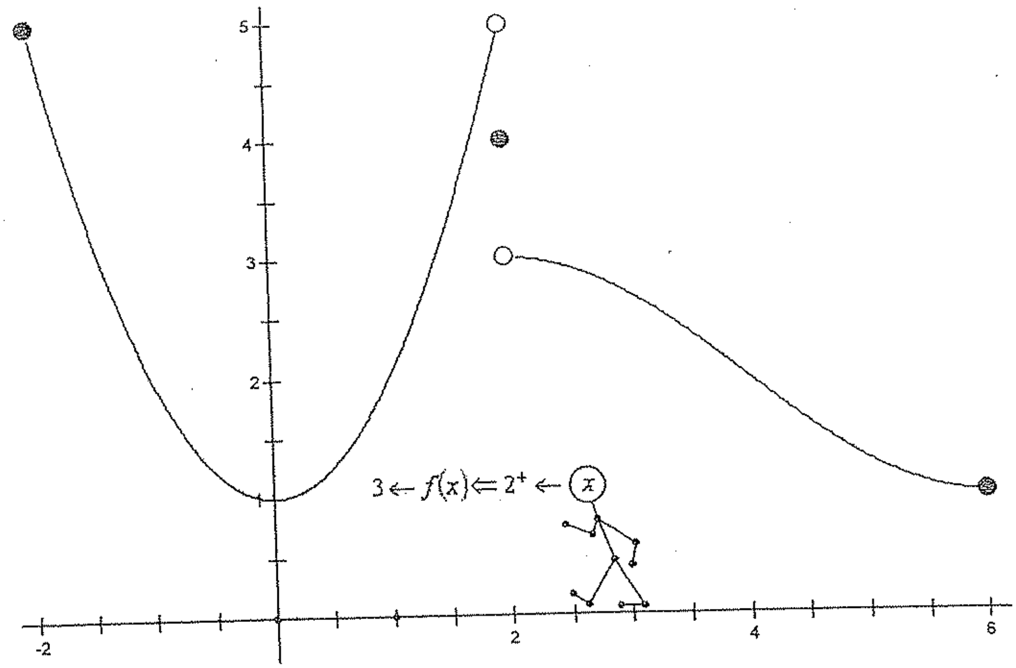

Figure 2

	Left Hand Limit If x approaches 2 from the left, then $f(x)$ approaches 5 or $x \to 2^- \Rightarrow f(x) \to 5$ or $\lim_{x \to 2^-} f(x) = 5$		**Right Hand Limit** If x approaches 2 from the right, then $f(x)$ approaches 3 or $x \to 2^+ \Rightarrow f(x) \to 3$ or $\lim_{x \to 2^+} f(x) = 3$	
	x	$f(x)$	x	$f(x)$
1	1.980	4.9204	2.97	2.7236
2	1.982	4.9283	2.89	2.7655
3	1.984	4.9363	2.80	2.8090
4	1.986	4.9442	2.72	2.8443
5	1.988	4.9521	2.63	2.8801
6	1.990	4.9601	2.55	2.9081
7	1.992	4.9681	2.46	2.9354
8	1.994	4.9760	2.38	2.9558
9	1.996	4.9840	2.29	2.9742
10	1.999	4.9960	2.21	2.9864
11	1.9998	4.9992	2.12	2.9956
12	1.99999	4.9999	2.04	2.9995
13	1.999999	4.99999	2.001	2.9999996
14	1.9999999	4.999999	2.0001	2.999999997
⋮	⋮	⋮	⋮	⋮
↓	↓	↓	↓	↓
∞	2^-	5	2^+	3

Figure 3

In our examples, we intuitively found $\lim_{x \to 2^-} f(x) = 5$ and $\lim_{x \to 2^+} f(x) = 3$. You may be asking yourself how do we really know that $L = 5, R = 3$ and not, say, $L = 5.000001, R = 3.0000001$. This point is well taken. We are only intuitively trying to understand the limit concept. We will soon develop criteria which will guarantee what the handed limits are, if they exist.

The Limit Itself

We assume that we have a function f that is defined near c (on both sides of c) but not necessarily at c; that is, $f(c)$ may not be defined. If $\lim_{x \to c^-} f(x) = L = R = \lim_{x \to c^+} f(x)$, we write $\lim_{x \to c} f(x) = L$, read "*The limit of* $f(x)$, *as x approaches c from either(both) side(s), is* L". The number L is called the **limit** of f at c.

Refer to Figure 4 and Figure 5 to see an example in which $\lim_{x \to 5} f(x) = 4$.

The Limit $\lim_{x \to 5} f(x) = 4$

$$f(x) = \begin{cases} \dfrac{\left|(x-2)^3 - 27\right|}{9} + 4 & x < 5 \\ 6 & x = 5 \\ \sqrt[3]{8x - 67} + 7 & x > 5 \end{cases}$$

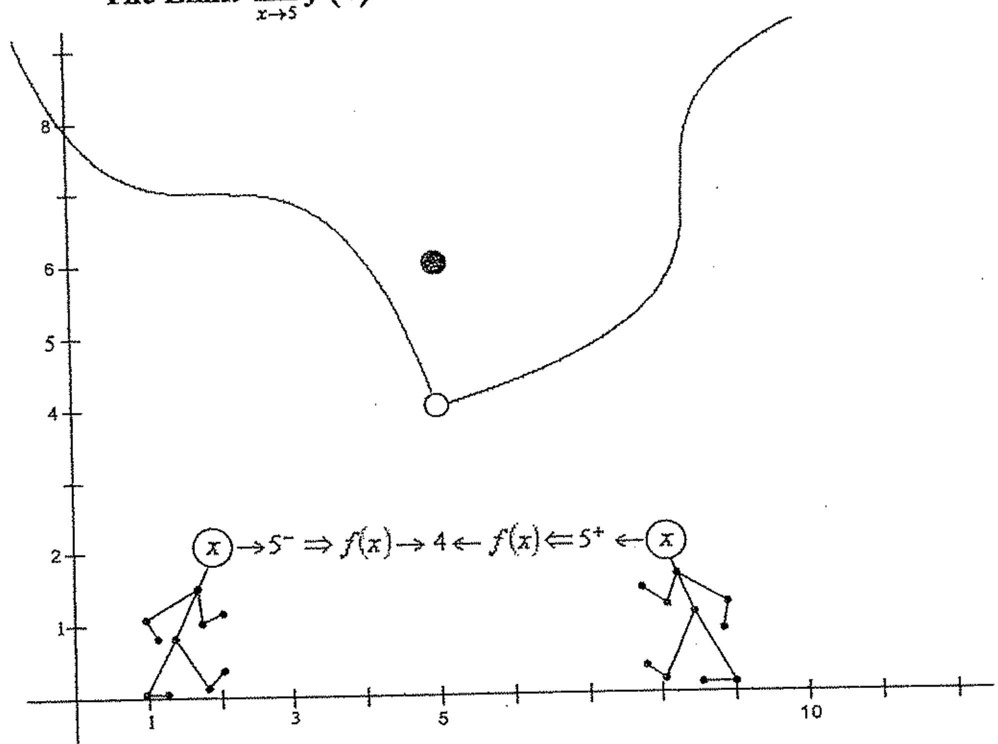

Figure 4

$\lim_{x\to 5} f(x)=4$				
Left Hand Limit			Right Hand Limit	
If x approaches 5 from the left, then $f(x)$ approaches 4 or $x\to 5^- \Rightarrow f(x)\to 4$ or $\lim_{x\to 5^-} f(x)=4$			If x approaches 5 from the right, then $f(x)$ approaches 4 or $x\to 5^+ \Rightarrow f(x)\to 4$ or $\lim_{x\to 5^+} f(x)=4$	
	x	$f(x)$	x	$f(x)$
1	4.44	5.3859	5.16	4.0482
2	4.51	5.2430	5.15	4.0451
3	4.57	5.1139	5.14	4.0421
4	4.63	4.9787	5.13	4.0390
5	4.70	4.8130	5.12	4.0360
6	4.76	4.6639	5.11	4.0330
7	4.82	4.5082	5.10	4.0299
8	4.89	4.3180	5.09	4.0269
9	4.91	4.2620	5.08	4.0239
10	4.93	4.2051	5.07	4.0209
11	4.98	4.0596	5.04	4.0119
12	4.99	4.0299	5.01	4.0030
13	4.999	4.0029	5.001	4.0002
14	4.9999	4.00029	5.0001	4.00002
15	4.99999	4.00003	5.00001	4.000002
⋮	⋮	⋮	⋮	⋮
↓	↓	↓	↓	↓
∞	5^-	4	5^+	4

Figure 5

Precise Definition of the Limit

We assume that we have a function f that is defined in a set of the form $(c-p,c)\cup(c,c+p)$, or, for those x's that satisfy $0<|x-c|<p$, some $p>0$. Now,

$\lim_{x\to c} f(x)=L$ means, for $x\neq c$,

If x approaches c, then $f(x)$ approaches L

or

$$x\to c \Rightarrow f(x)\to L$$

or

$$|x-c|\to 0 \Rightarrow |f(x)-L|\to 0$$

In the last implication, we are given that $0\neq|x-c|\to 0$ but how can we guarantee that $|f(x)-L|\to 0$? The answer to this last point is the essence of the limit concept. So, we must try to understand what it could mean for $|f(x)-L|\to 0$. Now, if $|f(x)-L|\to 0$, we will understand this to mean that $|f(x)-L|$ *can be made as small as we desire*. How can we verify that we can do this? We can do this by applying the following test:

Let ε (epsilon) be an arbitrary positive test number[†]. Then, our understanding that $|f(x)-L|$ *can be made as small as we desire* becomes the requirement that one can solve $|f(x)-L|<\varepsilon$ for x in the following sense:

We can produce a positive number δ (delta) no greater than p[‡] so that every $x \neq c$, with distance less than δ from c, satisfies the inequality $|f(x)-L|<\varepsilon$.

In summary, $\lim_{x \to c} f(x) = L$ means that given $\varepsilon > 0$ one can produce a positive number δ, with $\delta \le p$, that answers the following:

$$0 < |x-c| < \boxed{\delta = ?} \Rightarrow |f(x)-L| < \varepsilon$$

Example 1: Prove $\lim_{x \to 1}(3x+2) = 5$, using $\varepsilon - \delta$.

Proof:

Set ε -Test:

Let $\varepsilon > 0$. We must answer.

$$0 < |x-1| < \boxed{\delta = ?} \Rightarrow |(3x+2)-5| < \varepsilon$$

Determine *P*:

First, we ask

Within what distance *p* of *c* should we begin our analysis of the behavior of $f(x)$ for *x* near *c*?

For functions that are polynomials, we may take $p = 1$. This means that our analysis of the behavior of $f(x)$ for x near 1 assumes that, initially, x will satisfy

(1) $0 < |x-1| < p$

Discover δ :

Determine Relationship Between the Expressions $|x-c|$ & $|f(x)-L|$:

Since $|x-1|$ controls when $|(3x+2)-5| < \varepsilon$, we want to determine how the expressions $|x-1|$ and $|(3x+2)-5|$ are related. Here $|x-1|$ is a factor of $|(3x+2)-5|$. Namely,

(2) $|(3x+2)-5| = 3|x-1|$

† In the future, we will simply write "$\varepsilon > 0$" to mean "ε is an arbitrary positive test number".

‡ The number δ will always satisfy $0 < \delta \le p$.

Then use (2) To Extract δ :

We want $\left|(3x+2)-5\right|<\varepsilon$. From (2), it is sufficient to require $3|x-1|<\varepsilon$. But $3|x-1|<\varepsilon \Leftrightarrow |x-1|<\frac{\varepsilon}{3}$. This suggests that we take $\delta=\min\left(p,\frac{\varepsilon}{3}\right)$ because we have two restrictions on $|x-1|$, namely, $0<|x-1|<p$ & $|x-1|<\varepsilon/3$.

Verify δ :

Since $\delta=\min\left(p,\frac{\varepsilon}{3}\right)$,

(3) $\underline{\underline{0<|x-1|<\underbrace{\delta}_{\le \varepsilon/3}^{\le p}\Rightarrow \left|(3x+2)-5\right|}} \underset{\text{By (2)}}{=} 3|x-1| \underset{\text{Because }|x-1|<\delta}{\underline{\underline{<}}} 3\delta=3\frac{\varepsilon}{3}=\underline{\underline{\varepsilon}}$

Summarizing, using the underline parts of (3), we finally obtain, for $\delta=\min\left(p,\frac{\varepsilon}{3}\right)$, that

$$0<|x-1|<\delta \Rightarrow \left|(3x+2)-5\right|<\varepsilon \quad ◼$$

Example 2: Prove $\lim_{x\to 2} x^2=4$, using $\varepsilon-\delta$.

Proof:

Set ε -Test:

Let $\varepsilon>0$ be our test number. We must answer.

$$0<|x-2|<\boxed{\delta=?}\Rightarrow \left|x^2-4\right|<\varepsilon$$

Determine P:

First, we ask

Within what distance p of c should we begin our analysis of the behavior of $f(x)=x^2$ for x near c?

For functions that are polynomials, we may take $p=1$. This means that our analysis of the behavior of $f(x)=x^2$ for x near 2 assumes that, initially, x must satisfy

(1) $0<|x-2|<p$.

Determine δ :

Then, we ask

How are $|x-2|$ and $\left|x^2-4\right|$ related?

$$\left|x^2-4\right|=|x-2||x+2|$$

Then, we ask

How big can $|x|$ become?

$$|x|=\left|(x-2)+2\right|\le|x-2|+|2| \underset{\text{Since }0<|x-2|<p}{<} p+|2|=3$$

In summary, we have

(2) $0<|x-2|<p \Rightarrow |x|<3$

Then, we ask

How big can the other factor $|x+2|$ become?

$$|x+2| \le |x|+|2| \underset{\text{By (2)}}{<} (p+|2|)+|2| = p+2|2| = 5$$

In summary, we have

(3) $0<|x-2|<p \Rightarrow |x+2|<5$

Then, we ask

What does (3) imply about the size of $|x^2-4|$?

$$0<|x-2|<p \Rightarrow |x^2-4| = |x-2||x+2| \underset{\text{By (3)}}{<} 5|x-2|$$

In summary,

(4) $0<|x-2|<p \Rightarrow |x^2-4|<5|x-2|$

Then, we ask

Finally, what conditions are required on $|x-c|$ to guarantee $|x^2-4|<\varepsilon$?

Recall that we want $|x^2-4|<\varepsilon$. However, from (4), we see it is sufficient to have

$$5|x-2|<\varepsilon$$

But

(5) $5|x-2|<\varepsilon \Leftrightarrow |x-2|<\dfrac{\varepsilon}{5}$

From (4) and (5), we see that we have two conditions that $|x-2|$ must satisfy. That is, we must have

(6) $0<|x-2|<p$ and $|x-2|<\dfrac{\varepsilon}{5}$

This suggests that we require $0<|x-2|<\delta$, where $\delta=\min\left(p,\dfrac{\varepsilon}{5}\right)$.

Verify δ :

Show that $\delta=\min\left(p,\dfrac{\varepsilon}{5}\right)$ works.

(7) $\underline{\underline{0<|x-2|<\delta}}_{\substack{\le p\\ \le \varepsilon/5}} \Rightarrow 0<|x-2|<p \underset{\text{By (4)}}{\Rightarrow} \underline{\underline{|x^2-4|}} < 5|x-2| \underset{\text{Because } 0<|x-2|<\delta}{<} 5\delta \underset{\text{By def. of } \delta}{\le} 5\dfrac{\varepsilon}{5} = \underline{\underline{\varepsilon}}$

Summarizing (7), we finally obtain, for $\delta=\min\left(p,\dfrac{\varepsilon}{5}\right)$,

$$0<|x-2|<\delta \Rightarrow |x^2-4|<\varepsilon \quad \blacksquare$$

Example 3: Prove $\lim_{x \to c} x^2 = c^2$, using $\varepsilon - \delta$.

Proof:

Set ε -Test:

Let $\varepsilon > 0$ be our test number. We must answer.

$$0 < |x-c| < \boxed{\delta = ?} \Rightarrow |x^2 - c^2| < \varepsilon$$

Determine P:

First, we ask

Within what distance p of c should we begin our analysis of the behavior of $f(x) = x^2$ for x near c?

For functions that are polynomials, we may take $p = 1$. This means that our analysis of the behavior of $f(x) = x^2$ for x near c assumes that, initially, x must satisfy

(1) $0 < |x-c| < p$.

Determine δ :

Then, we ask

How are $|x-c|$ and $|x^2 - c^2|$ related?

$$|x^2 - c^2| = |x-c||x+c|$$

Then, we ask

How big can $|x|$ become?

$$|x| = |(x-c)+c| \leq |x-c| + |c| \underset{\text{Since } 0<|x-c|<p}{<} p + |c|$$

In summary, we have

(2) $0 < |x-c| < p \Rightarrow |x| < p + |c|$

Then, we ask

How big can the other factor $|x+c|$ become?

$$|x+c| \leq |x| + |c| \underset{\text{By (2)}}{<} (p+|c|) + |c| = p + 2|c| \equiv A$$

In summary, we have

(3) $0 < |x-c| < p \Rightarrow |x+c| < p + 2|c| \equiv A$

Then, we ask

What does (3) imply about the size of $|x^2 - c^2|$?

$$0 < |x-c| < p \Rightarrow |x^2 - c^2| = |x-c||x+c| \underset{\text{By (3)}}{<} |x-c|A$$

In summary,

(4) $0 < |x-c| < p \Rightarrow |x^2 - c^2| < |x-c|A$

Then, we ask

Finally, what conditions are required on $|x-c|$ to guarantee $|x^2 - c^2| < \varepsilon$?

Recall that we want $|x^2 - c^2| < \varepsilon$. However, from (4), we see it is sufficient to have

$$A|x-c|<\varepsilon$$

But

(5) $A|x-c|<\varepsilon \Leftrightarrow |x-c|<\dfrac{\varepsilon}{A}$

From (4) and (5), we see that we have two conditions that $|x-c|$ must satisfy. That is, we must have

(6) $0<|x-c|<p$ and $|x-c|<\dfrac{\varepsilon}{A}$

This suggests that we require $0<|x-c|<\delta$, where $\delta=\min\left(p,\dfrac{\varepsilon}{A}\right)$.

Verify δ :

Show that $\delta=\min\left(p,\dfrac{\varepsilon}{A}\right)$ works.

(7) $\underline{\underline{0<|x-c|<\delta}} \underset{\le\varepsilon/A}{\overset{\le p}{\Rightarrow}} 0<|x-c|<p \underset{\text{By (4)}}{\Rightarrow} \underline{\underline{|x^2-c^2|}}<|x-c|A \underset{\text{Because } 0<|x-c|<\delta}{<} \delta A \underset{\text{By def. of } \delta}{\le} \dfrac{\varepsilon}{A}A=\underline{\underline{\varepsilon}}$

Summarizing, using the underlined parts of (7), we finally obtain an answer to the original question posed above: We have, for $\delta=\min\left(p,\dfrac{\varepsilon}{A}\right)$, that

$$0<|x-c|<\delta \Rightarrow |x^2-c^2|<\varepsilon \quad ■$$

Example 4: Let $f(x)=\dfrac{1}{x}$. For $c\neq 0$, Prove $\lim\limits_{x\to c} f(x)=\dfrac{1}{c}$, using $\varepsilon-\delta$.

Proof:

Set ε -Test:

Let $\varepsilon>0$ be our test number. We must answer

(1) $$0<|x-c|<\boxed{\delta=?}\Rightarrow\left|\frac{1}{x}-\frac{1}{c}\right|<\varepsilon$$

Determine *P*:

First, we ask

Within what distance *p* of *c* should we begin our analysis of the behavior of $f(x)$ near *c*?

Since $x\neq 0$, we may use any value of p that defines an interval about c that does not include 0. For convenience, we take $p=|c|/2$. This means that our analysis of the behavior of $f(x)$ near c assumes that, initially, x must satisfy

(2) $0<|x-c|<p$

Determine δ :

Then, we ask

How are $|x-c|$ and $\left|\frac{1}{x}-\frac{1}{c}\right|$ related?

(3) $$\underline{\underline{\left|\frac{1}{x}-\frac{1}{c}\right|}}=\left|\frac{c-x}{xc}\right|=\left|\frac{c-x}{xc}\right|=\left|\frac{x-c}{xc}\right|=\frac{|x-c|}{|c||x|}=\underline{\underline{|x-c|\frac{1}{|c||x|}}}$$

Summarizing, using the underlined parts of (3), we have

(4) $$\left|\frac{1}{x}-\frac{1}{c}\right|=|x-c|\frac{1}{|c||x|}$$

Then, we ask

How small can $|x|$ become?

(5) $$\underline{\underline{|x|}}=|(x-c)+c|\geq\big||c|-|x-c|\big|\underline{\underline{\geq}}|c|-\frac{|c|}{2}=\underline{\underline{\frac{|c|}{2}}}$$

Summarizing, using the underlined parts of (5), we have

(6) $$0<|x-c|<p\Rightarrow|x|>\frac{|c|}{2}$$

Then, we ask

How big can the other factor $\frac{1}{|c||x|}$ in (3) become?

(7) $$\underline{\underline{\frac{1}{|c||x|}}}\underset{\text{By (6)}}{\underline{\underline{\leq}}}\frac{1}{|c|\frac{|c|}{2}}=\underline{\underline{\frac{2}{|c|^2}}}$$

Summarizing, using the underlined parts of (7), we have

(8) $$0<|x-c|<p\Rightarrow\frac{1}{|c||x|}<\frac{2}{|c|^2}$$

Then, we ask

What does (8) imply about the size of $\left|\frac{1}{x}-\frac{1}{c}\right|$?

(9) $$\underline{\underline{\left|\frac{1}{x}-\frac{1}{c}\right|}}\underset{\text{By (4)}}{=}|x-c|\frac{1}{|c||x|}\underset{\text{By item (8)}}{\underline{\underline{\leq}}}\underline{\underline{\frac{2|x-c|}{|c|^2}}}$$

Summarizing, using the underlined parts of (8), we have

(10) $$0<|x-c|<p\Rightarrow\left|\frac{1}{x}-\frac{1}{c}\right|<\frac{2|x-c|}{|c|^2}$$

Then, we ask

Finally, what requirements must $|x-c|$ satisfy to guarantee $\left|\frac{1}{x}-\frac{1}{c}\right|<\varepsilon$?

Recall that we want $\left|\frac{1}{x}-\frac{1}{c}\right|<\varepsilon$. However, from (10), we see it is sufficient to have $0<\frac{2|x-c|}{|c|^2}<\varepsilon$. But $\frac{2|x-c|}{|c|^2}<\varepsilon\Leftrightarrow 0<|x-c|<\frac{\varepsilon|c|^2}{2}$. Thus, we need

(11) $0<|x-c|<\frac{\varepsilon|c|^2}{2}$

Thus, from (10) and (11), we see that we have two requirements that $|x-c|$ must satisfy. That is, we must have $0<|x-c|<p$ and $0<|x-c|<\frac{\varepsilon|c|^2}{2}$. This suggests that we require $0<|x-c|<\delta$, where

(12) $\delta=\min\left(p,\frac{\varepsilon|c|^2}{2}\right)$.

Verify δ :

Show that $\delta=\min\left(p,\frac{\varepsilon|c|^2}{2}\right)$ works.

(13) $\underline{\underline{0<|x-c|<\delta}}\ {}^{\le p}_{\le\varepsilon|c|^2/2} \underset{=}{\Rightarrow} 0<|x-c|<p \Rightarrow \underline{\underline{\left|\frac{1}{x}-\frac{1}{c}\right|}} \underset{\text{By (10)}}{\le} \frac{2|x-c|}{|c|^2} \underset{\text{Because } 0<|x-c|<\delta}{<} \frac{2}{|c|^2}\cdot\delta \underset{\text{By def. of }\delta}{\le} \frac{2}{|c|^2}\cdot\frac{|c|^2}{2}\varepsilon=\underline{\underline{\varepsilon}}$

Summarizing, using the underlined parts of (13), we finally obtain an answer to the original question posed in (1): We have, for $\delta=\min\left(p,\frac{\varepsilon|c|^2}{2}\right)$, that

$$0<|x-c|<\delta \Rightarrow \left|\frac{1}{x}-\frac{1}{c}\right|<\varepsilon \quad \blacksquare$$

Example 5: For $c>0$, Prove $\lim_{x\to c}\sqrt{x}=\sqrt{c}$, using $\varepsilon-\delta$.

Proof:

Set ε -Test:

Let $\varepsilon>0$ be our test number. We must answer

$$0<|x-c|<\boxed{\delta=?}\Rightarrow\left|\sqrt{x}-\sqrt{c}\right|<\varepsilon$$

Determine *P*:

First, we ask

Within what distance *p* of *c* should we begin our analysis of the behavior of $f(x)=\sqrt{x}$ for x near c?

Since we must have $x>0$, we may choose any deleted interval about c, in which $x>0$. Thus, we may take $p=\frac{|c|}{2}=\frac{c}{2}$. This means that our analysis of the behavior of $f(x)=\sqrt{x}$ for x near c assumes that, initially, x must satisfy

(1) $0<|x-c|<p$.

Determine δ :

Then, we ask

How are $|x-c|$ and $|\sqrt{x}-\sqrt{c}|$ related?

$$|x-c| = |\sqrt{x}-\sqrt{c}||\sqrt{x}+\sqrt{c}|$$

So,

$$|\sqrt{x}-\sqrt{c}| = \frac{1}{|\sqrt{x}+\sqrt{c}|}|x-c| < \frac{1}{0+\sqrt{c}}|x-c| < \frac{1}{\sqrt{c}}|x-c|$$

Thus, in summary, we have

(2) $0<|x-c|<p \Rightarrow |\sqrt{x}-\sqrt{c}| < \frac{1}{\sqrt{c}}|x-c|$

Then, we ask

Finally, what conditions must be imposed on $|x-c|$ to guarantee $|\sqrt{x}-\sqrt{c}|<\varepsilon$?

Recall that we want $|\sqrt{x}-\sqrt{c}|<\varepsilon$. However, from (2), we see it is sufficient to have

$$\frac{1}{\sqrt{c}}|x-c|<\varepsilon$$

But

(3) $\frac{1}{\sqrt{c}}|x-c|<\varepsilon \Leftrightarrow |x-c|<\varepsilon\sqrt{c}$

From (2) and (3), we see that we have two conditions that $|x-c|$ must satisfy. That is, we must have

(4) $0<|x-c|<p$ and $|x-c|<\varepsilon\sqrt{c}$

This suggests that we require $0<|x-c|<\delta$, where $\delta=\min(p,\varepsilon\sqrt{c})$.

Verify δ :

Show that $\delta=\min(p,\varepsilon\sqrt{c})$ works.

(5) $\underline{\underline{0<|x-c|<\delta}} \underset{\le\varepsilon\sqrt{c}}{\overset{\le p}{\Rightarrow}} 0<|x-c|<p \underset{\text{By (2)}}{\Rightarrow} \underline{\underline{|\sqrt{x}-\sqrt{c}|}} < \frac{1}{\sqrt{c}}|x-c| \underset{\substack{\text{Since}\\ 0<|x-c|<\delta}}{<} \frac{1}{\sqrt{c}}\delta \underset{\text{By def. of }\delta}{\le} \frac{1}{\sqrt{c}}\varepsilon\sqrt{c} = \underline{\underline{\varepsilon}}$

Summarizing, using the underlined parts of (5), we finally obtain an answer to the original question posed above: That is, we have, for $\delta=\min(p,\varepsilon\sqrt{c})$, that

$$0<|x-c|<\delta \Rightarrow |\sqrt{x}-\sqrt{c}|<\varepsilon \quad ∎$$

Example 6: Let $f(x)=x^2+x+1$. Prove $\lim_{x\to 2} f(x)=7$, using $\varepsilon-\delta$.

Proof:

Set ε -Test:

Let $\varepsilon>0$ be our test number. We must answer

(1) $$0<|x-2|<\boxed{\delta=?}\Rightarrow |(x^2+x+1)-7|<\varepsilon$$

Determine *P*:

First, we ask

Within what distance *p* of 2 should we begin our analysis of the behavior of $f(x)=x^2+x+1$ near 2?

Since polynomials are defined everywhere, we may use any value of *p*. For convenience we take $p=1$. This means that our analysis of the behavior of $f(x)$ near 2 assumes that, initially, x must satisfy

(2) $0<|x-2|<p$.

Determine δ :

Then, we ask

How are $|x-2|$ and $|(x^2+x+1)-7|=|x^2+x-6|$ related?

(3) $|x^2+x-6|=|x-2||x+3|$

Then, we ask

How big can $|x|$ become?

(4) $\underline{\underline{|x|}}=|(x-2)+2|\le|x-2|+|2| \underset{\text{Since } 0<|x-2|<p}{\underline{\underline{<}}} p+|2|=1+2=\underline{\underline{3}}$

Summarizing, using the underlined parts of (4), we have

(5) $0<|x-2|<p\Rightarrow|x|<3$

Then, we ask

How big can the other factor $|x+3|$ in (3) become?

(6) $\underline{\underline{|x+3|}}\le|x|+|3| \underset{\text{By (5)}}{\underline{\underline{<}}} 3+|3|=3+3=\underline{\underline{6}}$

Summarizing, using the underlined parts of (6), we have

(7) $0<|x-2|<p\Rightarrow|x+3|<6$

Then, we ask

What does (7) imply about the size of $|(x^2+x+1)-7|$?

(8) $\underline{\underline{|(x^2+x+1)-7|}}=|x^2+x-6|=|x-2||x+3| \underset{\text{By (7)}}{\underline{\underline{<}}} \underline{\underline{6\cdot|x-2|}}$

Summarizing, using the underlined parts of (8), we have

(9) $0<|x-2|<p\Rightarrow|(x^2+x+1)-7|<6\cdot|x-2|$

Then, we ask

Finally, what requirements must $|x-2|$ satisfy to guarantee $|(x^2+x+1)-7|<\varepsilon$?

Recall that we want $|(x^2+x+1)-7|<\varepsilon$. However, from (9), we see it is sufficient to have $6\cdot|x-2|<\varepsilon$.

But $6\cdot|x-2|<\varepsilon \Leftrightarrow |x-2|<\frac{\varepsilon}{6}$. Thus, we also need

(10) $|x-2|<\frac{\varepsilon}{6}$

Thus, from (9) and (10), we see that we have two requirements that $|x-c|$ must satisfy. That is, we must have $0<|x-2|<p$ and $0<|x-2|<\frac{\varepsilon}{6}$. This suggests that we require $0<|x-2|<\delta$, where

(11) $\delta=\min\left(p,\frac{\varepsilon}{6}\right)$.

Verify δ :

Show that $\delta=\min\left(p,\frac{\varepsilon}{6}\right)$ works.

(12) $\underline{0<|x-2|<\delta} \underset{\leq\varepsilon/6}{\overset{\leq p}{\Rightarrow}} 0<|x-2|<p \underset{\text{By (9)}}{\Rightarrow} \underline{|(x^2+x+1)-7|} < 6\cdot|x-2| \underset{\text{Because } 0<|x-2|<\delta}{<} 6\cdot\delta \underset{\text{By def. of } \delta}{\leq} 6\cdot\frac{\varepsilon}{6}=\underline{\varepsilon}$

Summarizing, using the underlined part of (12), we finally obtain an answer to the original question posed in (1), for $\delta=\min\left(p,\frac{\varepsilon}{6}\right)$,

$$0<|x-2|<\delta \Rightarrow |(x^2+x+1)-7|<\varepsilon \quad \blacksquare$$

Homework. Let $f(x)=\dfrac{\sin(2x)}{x}$. Use your calculator and intuition to determine $\lim_{x\to 0} f(x)$ by filling in the following table.

$\lim_{x\to 0} f(x)=?$				
Left Hand Limit			Right Hand Limit	
$\lim_{x\to 0^-} f(x)=?$			$\lim_{x\to 0^+} f(x)=?$	
	x	$f(x)$	x	$f(x)$
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				
17				
18				
19				
20				
⋮	⋮	⋮	⋮	⋮
↓	↓	↓	↓	↓
∞	0^-	?	0^+	?

Homework. Let $f(x)=\dfrac{e^x-1}{x}$. Use your calculator and intuition to determine $\lim\limits_{x\to 0} f(x)$ by filling in the following table.

$\lim\limits_{x\to 0} f(x)=?$				
	Left Hand Limit		Right Hand Limit	
	$\lim\limits_{x\to 0^-} f(x)=?$		$\lim\limits_{x\to 0^+} f(x)=?$	
	x	$f(x)$	x	$f(x)$
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				
17				
18				
19				
20				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
∞	0^-	?	0^+	?