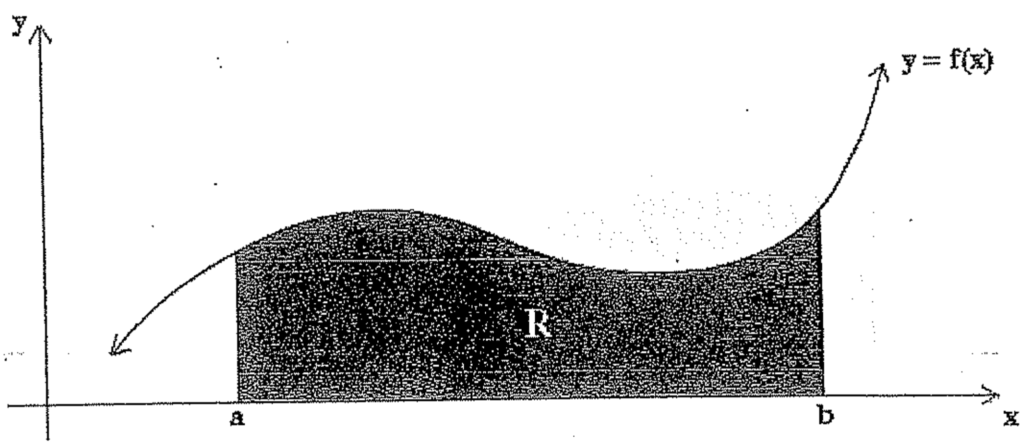
The Definite Integral

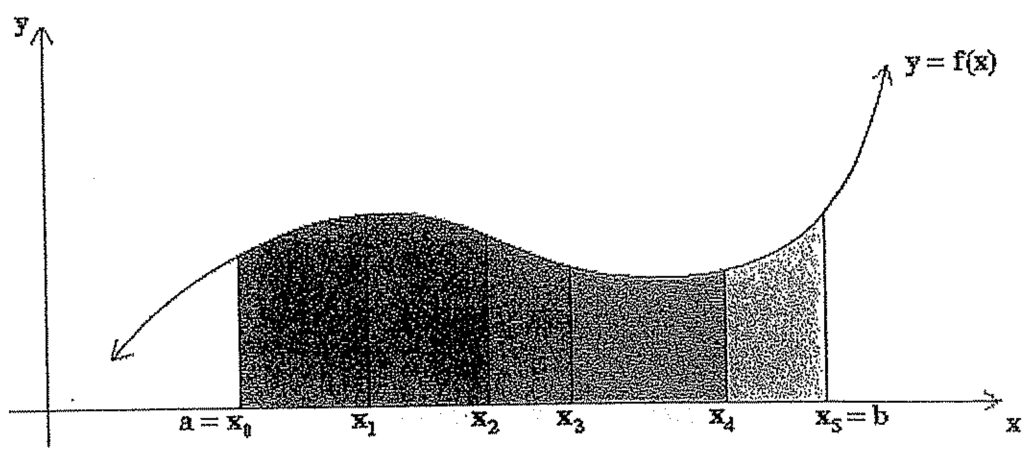
The Riemann Integral

What is area? We are all familiar with determining the area of simple geometric figures such as rectangles and triangles. However, how do we determine the area of a region R whose boundary may consist of non rectilinear curves, such as a parabola? To see how this could be done let us consider the following process.

Suppose that a function f is continuous and non-negative on an interval $[a,b]$. We wish to know what it means to compute the area of the region R bounded above by the curve $y = f(x)$, below by the x-axis, and, on the sides, by the lines $x = a$ and $x = b$, in short, the area under the curve $y = f(x)$, as seen in the figure below.



We will obtain the area of the region R as the limit of a sum of areas of rectangles as follows: First, we divide the interval $[a,b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. The intervals need not all be the same length. Let the lengths of these intervals be Δx_1, Δx_2, $\ldots$, Δx_n, respectively. This process divides the region R into n strips (see the figure below).

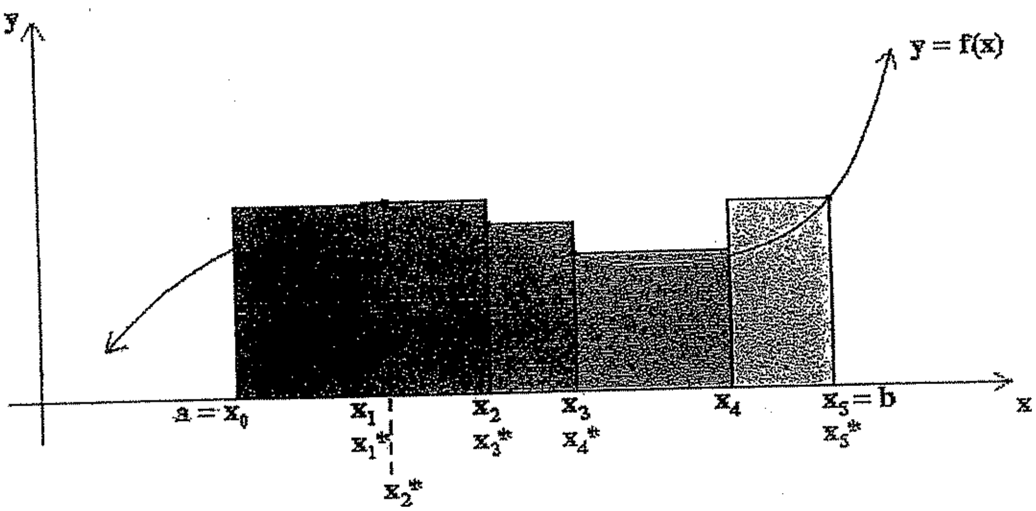


let's approximate each strip by a rectangle with height equal to the height of the curve $y = f(x)$ at some arbitrary point in the erval. That is, for the first subinterval $[x_0, x_1]$ select some x_1^* contained in that subinterval and use $f(x_1^*)$as the height of the ectangle. The area of that rectangle is then $f(x_1^*)\Delta x_1$.

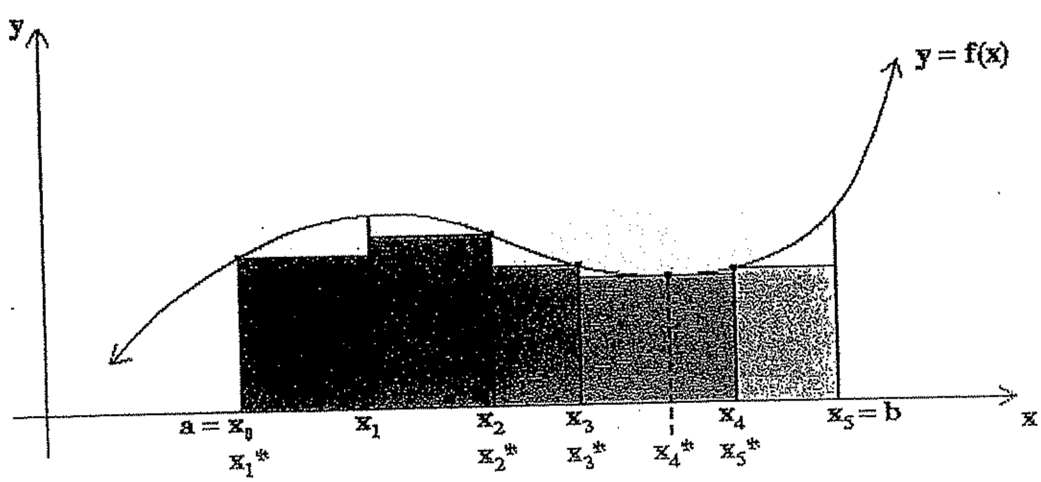
arly, for each remaining subinterval $[x_{k-1}, x_k]$, $2 \le k \le n$,], we will choose some x_k^* and calculate the area of the corresponding ıgle to be $f(x_k^*)\Delta x_k$. The approximate area of the region R is then the sum of these rectangular areas, denoted by

$$) = \sum_{k=1}^{n} f(x_k^*) \cdot \Delta x_k .$$

nding on what points we select for the x_k^*'s, our estimate may be too large or too small. For example, if we choose each x_i^* to be oint in its subinterval giving the maximum height, we will overestimate the area of R, called the Upper Sum (see the figure w).



the other hand, if we choose each x_k^* to be the point in its subinterval giving the minimum height, we will underestimate the area ₹, called the Lower Sum (see the figure below).

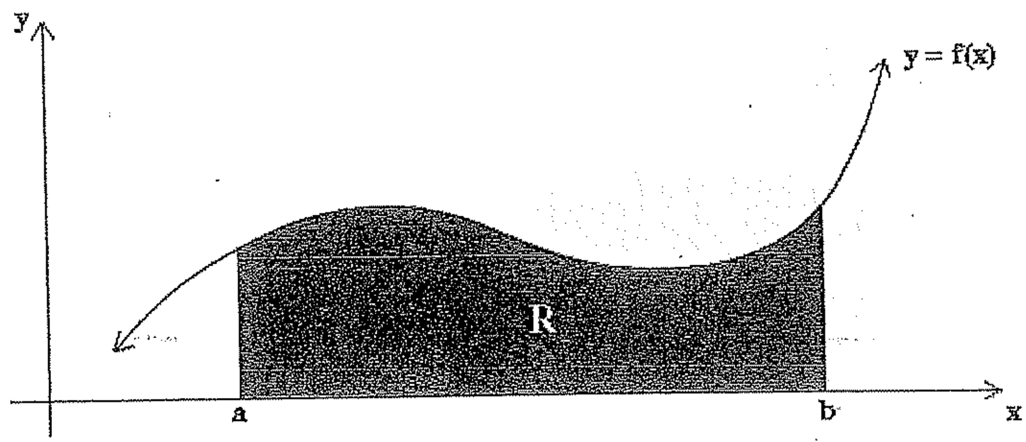


The Definite Integral

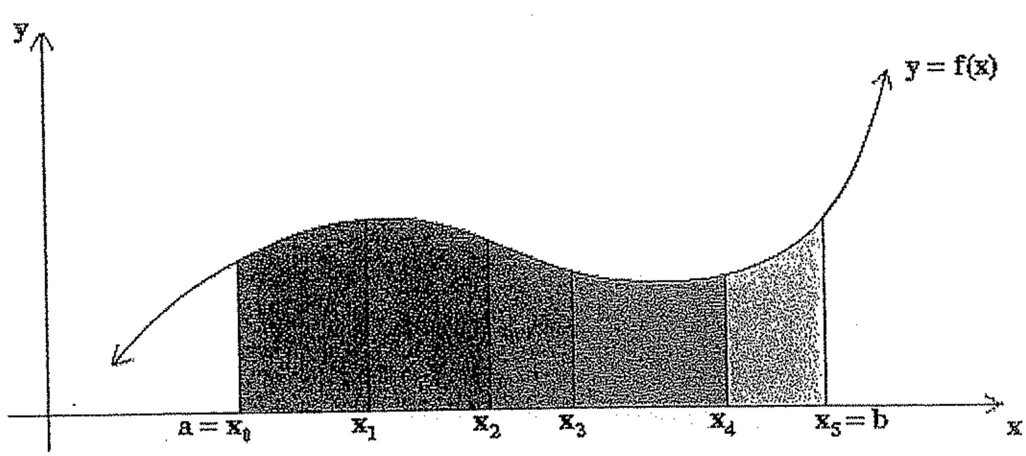
The Riemann Integral

What is area? We are all familiar with determining the area of simple geometric figures such as rectangles and triangles. However, how do we determine the area of a region R whose boundary may consist of non rectilinear curves, such as a parabola? To see how this could be done let us consider the following process.

Suppose that a function f is continuous and non-negative on an interval $[a, b]$. We wish to know what it means to compute the area of the region R bounded above by the curve $y = f(x)$, below by the x-axis, and, on the sides, by the lines $x = a$ and $x = b$, in short, the area under the curve $y = f(x)$, as seen in the figure below.



We will obtain the area of the region R as the limit of a sum of areas of rectangles as follows: First, we divide the interval $[a, b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{n-1}, x_n]$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. The intervals need not all be the same length. Let the lengths of these intervals be Δx_1, Δx_2, ..., Δx_n, respectively. This process divides the region R into n strips (see the figure below).

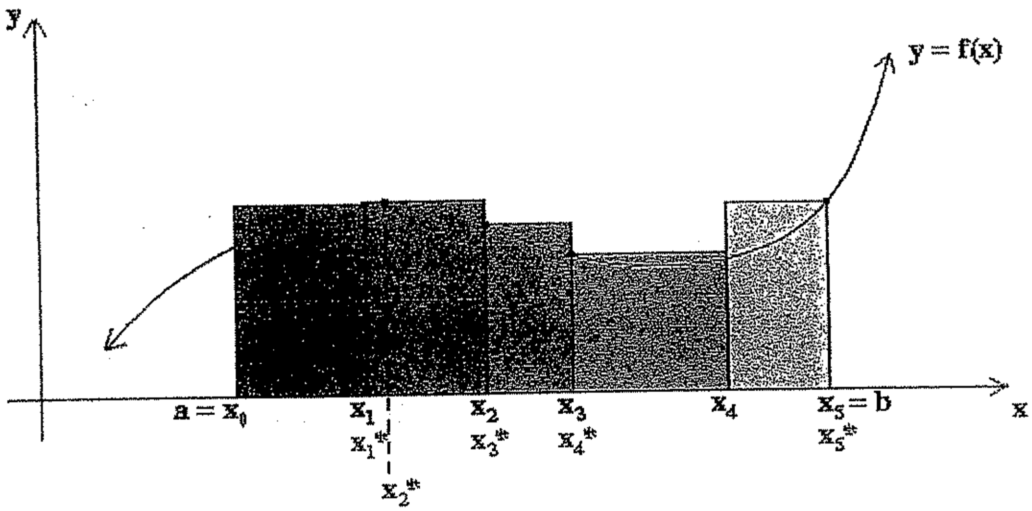


t, let's approximate each strip by a rectangle with height equal to the height of the curve $y = f(x)$ at some arbitrary point in the nterval. That is, for the first subinterval $[x_0, x_1]$ select some x_1^* contained in that subinterval and use $f(x_1^*)$ as the height of the rectangle. The area of that rectangle is then $f(x_1^*)\Delta x_1$.

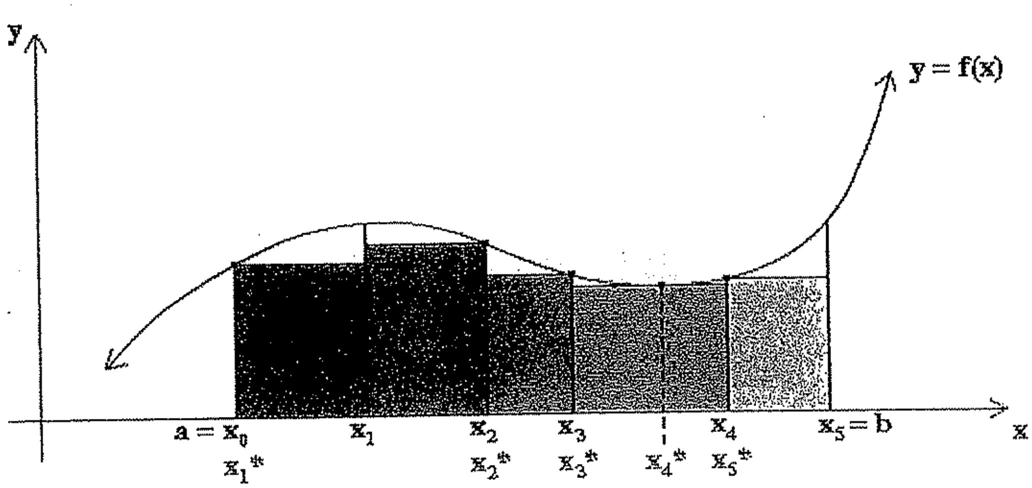
ilarly, for each remaining subinterval $[x_{k-1}, x_k]$, $2 \le k \le n$,], we will choose some x_k^* and calculate the area of the corresponding angle to be $f(x_k^*)\Delta x_k$. The approximate area of the region R is then the sum of these rectangular areas, denoted by

$$P) = \sum_{k=1}^{n} f(x_k^*) \Delta x_k .$$

ending on what points we select for the x_k^*'s, our estimate may be too large or too small. For example, if we choose each x_i^* to be point in its subinterval giving the maximum height, we will overestimate the area of R, called the Upper Sum (see the figure ow).



n the other hand, if we choose each x_k^* to be the point in its subinterval giving the minimum height, we will underestimate the area R, called the Lower Sum (see the figure below).



Now, if the sum $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$ approaches a limit as the length of the subintervals $[x_{k-1}, x_k]$ approach zero, regardless of the starred points x_k^* chosen, we then define the area of the region R to be precisely this limit. Note the beauty of this definition. Since we really do not know what area really is, we let our intuition develop a process that we legitimize as the analytic meaning of area under a continuous curve. We will now formalize this process in the following development, called the Riemann Integral.

Riemann Sums

Def: P is said to be a **partition** of the closed interval $[a,b]$ if P is a finite subset of $[a,b]$ which contains both a and b.

You may index the elements of P so that if $P = \{x_0 = a, x_1, x_2, x_3, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}, x_n = b\} \subseteq [a,b]$, you may conclude that $x_0 = a < x_1 < x_2 < \ldots < x_{k-1} < x_k < \ldots < x_{n-1} < x_n = b$.

Def: Let $P = \{x_0 = a, x_1, x_2, x_3, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}, x_n = b\} \subseteq [a,b]$ be a partition of $[a,b]$. We define

- $\Delta x_k = x_k - x_{k-1}, 1 \le k \le n$, called the **width** of the subinterval $[x_{k-1}, x_k]$
- $\|P\| = \max_{1 \le k \le n} \Delta x_k$, called the **norm** or **mesh** of the partition P.

Def: A partition P is said to be a **regular** partition of $[a,b]$ if, for some $n \in \mathbb{N}$, $\Delta x_k = \frac{(b-a)}{n}$, $1 \le k \le n$. In this case, $x_k = a + \frac{(b-a)}{n}k$, $0 \le k \le n$.

Def: Let f be a function bounded on $[a,b]$ and $P = \{x_0 = a, x_1, x_2, x_3, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}, x_n = b\}$ a partition of $[a,b]$. By a **Riemann sum** of f over $[a,b]$, we mean the sum represented by $S_f^*(P)$[1] and defined by

$$S_f^*(P) \overset{\text{def}}{=} \sum_{k=1}^{n} f(x_k^*)\Delta x_k \text{, any choice of } x_k^* \in [x_{k-1}, x_k],\ 1 \le k \le n.$$

Remark: If $f \ge 0$ on $[a,b]$, then $S_f^*(P)$ can be thought of as an approximation to the area under the curve of f from $x = a$ to $x = b$.

Def: Let f be a function bounded on $[a,b]$. We say that f is **Riemann integrable** over $[a,b]$ if there exists a real number I such that $\forall \varepsilon > 0\ \exists \delta > 0 \ni$, for every partition $P = \{x_0 = a, x_1, x_2, x_3, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}, x_n = b\}$ of $[a,b]$ and any choice of $x_k^* \in [x_{k-1}, x_k]$, $1 \le k \le n$, we have

$$\|P\| < \delta \Rightarrow |S_f^*(P) - I| < \varepsilon$$ [2]

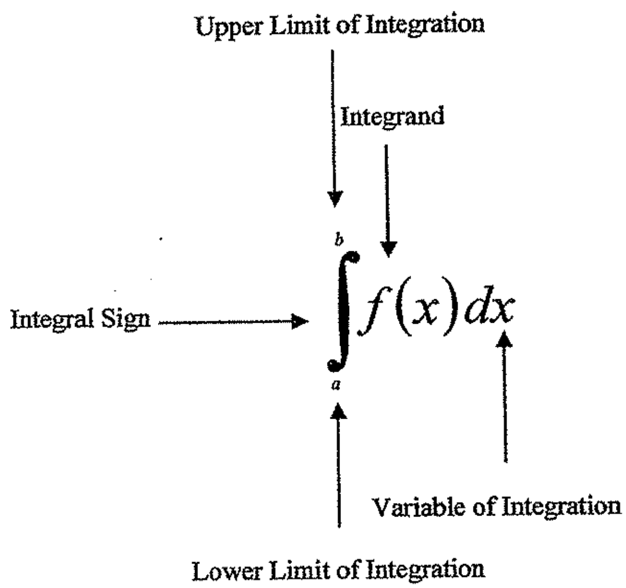
---

[1] Purcell writes R_P for $S_f^*(P)$.

[2] The notation $S_f^*(P)$ for the Riemann sum in this limit definition means that it does not matter what x_k^*'s are chosen. We only require that $x_k^* \in [x_{k-1}, x_k]$, $1 \le k \le n$.

n this case, we say $\lim_{\|P\|\to 0} S_f^*(P)$ exists, write $I = \lim_{\|P\|\to 0} S_f^*(P)$ and denote I by $\int_a^b f$ or $\int_a^b f(x)\,dx$, called the **definite/Riemann ntegral** of f over $[a,b]$.

The components that make up the Definite/Riemann Integral are named as follows:



Remark: The limit $\lim_{\|P\|\to 0} S_f^*(P)$ satisfies all the usual properties of limits.

Remark: For $x_k^* \in [x_{k-1}, x_k]$, it is convenient sometimes to use one of the following:

1. $x_k^* = a + \frac{(b-a)}{n}k$, $0 \le k \le n-1$. The left end points of the regular partition P_n.
2. $x_k^* = a + \frac{(b-a)}{n}k$, $1 \le k \le n$. The right end points of the regular partition P_n.
3. $x_k^* = \frac{x_k + x_{k-1}}{2}$, $1 \le k \le n$. The midpoints of each subinterval $[x_{k-1}, x_k]$, $1 \le k \le n$.
4. $x_k^* \in [x_{k-1}, x_k], 1 \le k \le n$, $\ni f(x_k^*)$ is a maximum in $[x_{k-1}, x_k]$, assuming f is continuous on $[a,b]$.
   The corresponding Riemann sum is then denoted by $U_f(P)$, called the **upper Darboux sum** of f over $[a,b]$[3].
5. $x_k^* \in [x_{k-1}, x_k], 1 \le k \le n$, $\ni f(x_k^*)$ is a minimum in $[x_{k-1}, x_k]$, assuming f is continuous on $[a,b]$.
   The corresponding Riemann sum is then denoted by $L_f(P)$, called the **lower Darboux sum** of f over $[a,b]$[4]

---

[3] If f is not continuous on $[a,b]$, the upper Darboux sum $U_f(P) = \sum_{k=1}^{n} M_k\,\Delta x_k$, where $M_k = \operatorname*{lub}_{x\in[x_{k-1},x_k]} f(x)$. See a previous handout on the real numbers, which discusses the least upper bound property.

[4] If f is not continuous on $[a,b]$, the lower Darboux sum $L_f(P) = \sum_{k=1}^{n} m_k\,\Delta x_k$, where $m_k = \operatorname*{glb}_{x\in[x_{k-1},x_k]} f(x)$. See a previous handout on the real numbers, which discusses the greatest lower bound property.

Note that we always have $L_f(P) \le S_f^*(P) \le U_f(P)$.

We now state the major theorem on the existence of the Riemann integral. We shall not prove this theorem since the proof involves ideas and techniques that are covered in a more advanced course.

Integrability Theorem: Let f be piecewise continuous [5] on $[a,b]$. Then f is integrable over $[a,b]$; that is, $\int_a^b f$ exists. In particular, if f is continuous on $[a,b]$, f is integrable over $[a,b]$.

Using the Integrability Theorem, we clearly have the following theorem.

Regular Integrability Theorem: Let f be continuous on $[a,b]$ and let P_n be the regular partition that corresponds to $n \in \mathbb{N}$, where $\Delta x_k = \frac{(b-a)}{n}$. Then, for any choice of $x_k^* \in [x_{k-1}, x_k]$, $1 \le k \le n$,

$$\int_a^b f = \lim_{n\to\infty} S_f^*(P_n) = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k .$$

Some conditions that are equivalent to Riemann integrability are given in the following theorem.

Riemann Condition Theorem: Let f be bounded on $[a,b]$. The following are equivalent.

1. f is Riemann integrable over $[a,b]$.
2. $\forall \varepsilon > 0 \exists P \text{ on } [a,b] \ni 0 \le U_f(P) - L_f(P) < \varepsilon$.
3. $\forall \varepsilon > 0 \exists \delta > 0 \ni, \forall P \text{ on } [a,b], \|P\| < \delta \Rightarrow 0 \le U_f(P) - L_f(P) < \varepsilon$.
4. If $S_{f,1}^*(P) = \sum_{k=1} f(x_{k,1}^*)\Delta x_k$ and $S_{f,2}^*(P) = \sum_{k=1} f(x_{k,2}^*)\Delta x_k$, where $x_{k,1}^*, x_{k,2}^* \in [x_{k-1}, x_k]$, then $\forall \varepsilon > 0 \exists P \text{ on } [a,b] \ni \left| S_{f,1}^*(P) - S_{f,2}^*(P) \right| < \varepsilon$, regardless of the chosen $x_{k,1}^*, x_{k,2}^* \in [x_{k-1}, x_k]$.
5. If $S_{f,1}^*(P) = \sum_{k=1}^{n} f(x_{k,1}^*)\Delta x_k$ and $S_{f,2}^*(P) = \sum_{k=1}^{n} f(x_{k,2}^*)\Delta x_k$, where $x_{k,1}^*, x_{k,2}^* \in [x_{k-1}, x_k]$, then $\forall \varepsilon > 0 \exists \delta > 0 \ni, \forall P \text{ on } [a,b], \|P\| < \delta \Rightarrow \left| S_{f,1}^*(P) - S_{f,2}^*(P) \right| < \varepsilon$, regardless of the chosen $x_{k,1}^*, x_{k,2}^* \in [x_{k-1}, x_k]$.

---

[5] A function f is said to be **piecewise continuous** on $[a,b]$ if f is continuous on $[a,b]$ except, possibly, at a finite number of points.

Example: Not all functions are Riemann integrable. Consider $f(x)=\begin{cases}0; & \text{if } x \text{ is irrational}\\ 1; & \text{if } x \text{ is rational}\end{cases}$ on $[0,1]$, called the **Dirichlet** function. Let $P=\{x_0=0,x_1,x_2,\ldots,x_n=1\}$ be a partition of the interval $[0,1]$. Now,

1. Since in any interval $[x_{k-1},x_k]$ there is an irrational number, we have $m_k = \min\limits_{x\in[x_{k-1},x_k]} f(x)=0$. Therefore,

$$L_f(P)=\sum_{k=1}^{n} m_k\,\Delta x_k = 0.$$

2. Since in any interval $[x_{k-1},x_k]$ there is a rational number, we have $M_k = \max\limits_{x\in[x_{k-1},x_k]} f(x)=1$. Therefore,

$$U_f(P)=\sum_{k=1}^{n} M_k\,\Delta x_k = 1.$$

Thus, for the Dirichlet function, we have shown that $L_f(P)=0$ and $U_f(P)=1$, for any partition P of $[0,1]$. This says that f is not Riemann integrable on $[0,1]$.

The following theorem shows us that not all functions have to be continuous in order to be integrable.

Theorem: Every $f\uparrow[a,b]$ is Riemann integrable on $[a,b]$. Let $P=\{x_0\equiv a,x_1,\cdots,x_{k-1},x_k,\cdots,x_{n-1},x_n\equiv b\}$ be any partition of $[a,b]$. Then, $L_f(P)=\sum_{k=1}^{n} f(x_{k-1})\Delta x_k$ and $U_f(P)=\sum_{k=1}^{n} f(x_k)\Delta x_k$. We then have

$$0\le U_f(P)-L_f(P)=\sum_{k=1}^{n} f(x_k)\Delta x_k-\sum_{k=1}^{n} f(x_{k-1})\Delta x_k=\sum_{k=1}^{n}\{f(x_k)-f(x_{k-1})\}\Delta x_k\le$$

$$\sum_{k=1}^{n}\{f(x_k)-f(x_{k-1})\}\|P\|=\|P\|\sum_{k=1}^{n}\{f(x_k)-f(x_{k-1})\}=\|P\|(f(b)-f(a)).$$

In summary, we have $0\le U_f(P)-L_f(P)\le\|P\|(f(b)-f(a))$, which implies f is Riemann integrable over $[a,b]$. ■

Example: Using the above definition of the Riemann integral, show that the area under the curve $f(x)=x^m$ form $x=a$ to $x=b$, where $m\in\mathbb{N}$ and $0\le a<b$. That is, show $\int_a^b x^m\,dx=\frac{b^{m+1}}{m+1}-\frac{a^{m+1}}{m+1}$.

Proof: First let us consider some preliminary remarks so that the proof is easier to understand. Recall, for $m\in\mathbb{N}$,

$$B^{m+1}-A^{m+1}=\left(B^m+B^{m-1}A+B^{m-2}A^2+\cdots+B^{m-j}A^j+\cdots+B^2A^{m-2}+BA^{m-1}+A^m\right)(B-A)=\left(\sum_{j=0}^{m}B^{m-j}A^j\right)(B-A)$$

If $A\le B$, then $A^m\le B^{m-j}A^j, 0\le j\le m$.

We now begin our proof.

Let $P = \{x_0 \equiv a, x_1, \cdots, x_{k-1}, x_k, \cdots, x_{n-1}, x_n \equiv b\}$ be any partition of $[a,b]$. In what follows think of x_k as B and x_{k-1} as A, where appropriate. Since $f(x) = x^m \uparrow$ on $[a,b]$, then, for $1 \le k \le n$, we have

$$(m+1)x_{k-1}^m \Delta x_k = \left(\sum_{j=0}^{m} x_{k-1}^m\right)\Delta x_k \le \left(\sum_{j=0}^{m} x_k^{m-j} x_{k-1}^j\right)\Delta x_k \le \left(\sum_{j=0}^{m} x_k^m\right)\Delta x_k = (m+1)x_k^m \Delta x_k$$

But $x_k^{m+1} - x_{k-1}^{m+1} = \left(\sum_{j=0}^{m} x_k^{m-j} x_{k-1}^j\right)\Delta x_k$. So,

(*) $$(m+1)x_{k-1}^m \Delta x_k \le x_k^{m+1} - x_{k-1}^{m+1} \le (m+1)x_k^m \Delta x_k .$$

Observe that,

(**) $$\sum_{k=1}^{n}\left(x_k^{m+1} - x_{k-1}^{m+1}\right) = \left(x_1^{m+1} - x_0^{m+1}\right) + \left(x_2^{m+1} - x_1^{m+1}\right) + \cdots + \left(x_{n-1}^{m+1} - x_{n-2}^{m+1}\right) + \left(x_n^{m+1} - x_{n-1}^{m+1}\right) = x_n^{m+1} - x_0^{m+1} = b^{m+1} - a^{m+1}.$$

This series is called a **telescoping series** because of the way the terms in the sum cancel.

Summing (*) from $k = 1$ to $k = n$, we obtain

(***) $$(m+1)\sum_{k=1}^{n} x_{k-1}^m \Delta x_k \le \sum_{k=1}^{n}\left(x_k^{m+1} - x_{k-1}^{m+1}\right) \le (m+1)\sum_{k=1}^{n} x_k^m \Delta x_k$$

Using our observation in (**) and dividing by $m+1$ in (***), we finally obtain from (***)

$$L_{x^m}(P) = \sum_{k=1}^{n} x_{k-1}^m \Delta x_k \le \frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1} \le \sum_{k=1}^{n} x_k^m \Delta x_k = U_{x^m}(P), \ \forall P \text{ of } [a,b].$$

Since $L_{x^m}(P) \le S_{x^m}^*(P) \le U_{x^m}(P)$,

$$\left|S_{x^m}^*(P) - \left(\frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1}\right)\right| \le U_{x^m}(P) - L_{x^m}(P) =$$

$$\sum_{k=1}^{n} x_k^m \Delta x_k - \sum_{k=1}^{n} x_{k-1}^m \Delta x_k = \sum_{k=1}^{n}\left(x_k^m - x_{k-1}^m\right)\Delta x_k \le$$

$$\|P\|\sum_{k=1}^{n}\left(x_k^m - x_{k-1}^m\right) = \|P\|\left(b^m - a^m\right)$$

Thus,

$$\left|S_{x^m}^*(P) - \left(\frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1}\right)\right| \le \|P\|\left(b^m - a^m\right)$$

his clearly implies that

$$\int_a^b x^m\,dx = \lim_{\|P\|\to 0} S^*(P) = \left(\frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1}\right). \blacksquare$$

Exercises: Let $f(x)=x^m$, $a,b\in\mathbb{R}$. Use the procedure in the above example to determine 1)-3).

1) $\int_a^b x^m\,dx$ if $a<b\le 0$ and m is even.

2) $\int_a^b x^m\,dx$ if $a<b\le 0$ and m is odd.

3) $\int_a^b x^m\,dx$ if $a<b$.

Def: $F(x)\Big|_a^b \overset{\text{def}}{=} F(b)-F(a)$ or $F(x)\Big|_{x=a}^{x=b} \overset{\text{def}}{=} F(b)-F(a)$.

Def: $F(x)\Big|_a \overset{\text{def}}{=} F(a)$ or $F(x)\Big|_{x=a} \overset{\text{def}}{=} F(a)$.

Rewriting our last example using our new notation, we obtain $\int_a^b x^m\,dx = \frac{x^{m+1}}{m+1}\Big|_a^b$. If we look carefully at the result of this example of determining area under a graph, we are led to the interesting observation that there seems to be a relationship between the process of definite integration, which is just a fancy way of performing sums, and the process of differentiation. That is, we see that an antiderivative of the integrand x^m is $\frac{x^{m+1}}{m+1}$ and the derivative of $\frac{x^{m+1}}{m+1}$ is x^m . This is no accident. We will soon develop a theorem that generalizes this relationship to any continuous integrand f over $[a,b]$. This theorem is called the Fundamental Theorem of Integral Calculus.

The proof of the Fundamental Theorem of Integral Calculus will be divided into two parts. The first part, called the First Fundamental theorem of Integral Calculus, shows us how to differentiate a variable integral.

The second part, called the Second Fundamental Theorem of Integral Calculus, shows us that one can compute the definite integral of a continous function by using any one of its antiderivatives. This part of the theorem has many practical applications, because it tremendously simplifies the computation of definite integrals.

The first published statement and proof of a restricted version of the Fundamental Theorem of Integral Calculus was given by James Gregory (1638–1675). Isaac Barrow (1630–1677) proved the first completely general version of this theorem. Barrow's student Sir Isaac Newton (1643–1727) completed the development of the surrounding mathematical theory, while Gottfried Leibniz (1646–1716) systematized the knowledge into a calculus involving infinitesimal quantities.

Lemma: Let $A, B \in \mathbb{R}$. If $\forall \varepsilon > 0$

a) $A + \varepsilon > B \Rightarrow A \geq B$.

b) $A > B - \varepsilon \Rightarrow A \geq B$.

Proof of 1: By contradiction. Assume $A < B$. Then, $B - A > 0$. Therefore, let $\varepsilon = (B - A)/2$. By hypothesis, $A + \varepsilon > B \Rightarrow A + (B - A)/2 > B \Rightarrow 2A + (B - A) > 2B \Rightarrow A > B$. This is a contradiction. ∎

Proof of 2: By hypothesis, $A > B - \varepsilon \Rightarrow A + \varepsilon > B \Rightarrow A \geq B$, by a).

Properties of the Riemann Integral

Let f, g be continuous on an interval $I \supseteq [a, b]$, $m = \min\limits_{x \in [a,b]} f(x)$ and $M = \max\limits_{x \in [a,b]} f(x)$. Then,

1. $m(b-a) \leq \int_a^b f \leq M(b-a)$, $\Leftarrow$ **Max-Min Rule.**

   Proof: Since $\int_a^b f = \lim\limits_{\|P\| \to 0} S_f^*(P)$, given $\varepsilon > 0 \, \exists P$ on $[a, b] \ni$

   a) $S_f^*(P) - \varepsilon < \int_a^b f < S_f^*(P) + \varepsilon$.

   Also,

   b) $m(b-a) \leq S_f^*(P) \leq M(b-a)$.

   Combining a) and b), we obtain $m(b-a) - \varepsilon \leq S_f^*(P) - \varepsilon < \int_a^b f < S_f^*(P) + \varepsilon \leq M(b-a) + \varepsilon$. In summary, we have

   $m(b-a) - \varepsilon < \int_a^b f < M(b-a) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have, by the above lemma applied to $m(b-a) - \varepsilon < \int_a^b f$ and to $\int_a^b f < M(b-a) + \varepsilon$, that $m(b-a) \leq \int_a^b f \leq M(b-a)$. ∎

2. $\alpha < f(x) < \beta \Rightarrow \alpha(b-a) < \int_a^b f < \beta(b-a)$ $\Leftarrow$ **Boundedness Rule.**

   Proof: $\alpha(b-a) < m(b-a) \leq \int_a^b f \leq M(b-a) < \beta(b-a) \Rightarrow \alpha(b-a) < \int_a^b f < \beta(b-a)$.

3. $f = c \Rightarrow \int_a^b f = c(b-a)$ $\Leftarrow$ **Constant Rule.**

4. $f \geq 0 \Rightarrow \int_a^b f \geq 0 \Leftarrow$ **Non-Negative Rule.**

Proof: By the Max-Min Rule, $\int_a^b f \geq m(b-a) \geq 0 \Rightarrow \int_a^b f \geq 0$. ∎

5. $f > 0 \Rightarrow \int_a^b f > 0 \Leftarrow$ **Positive Rule.**

Proof: By the Max-Min Rule, $\int_a^b f \geq m(b-a) > 0 \Rightarrow \int_a^b f > 0$.

6. $\int_a^b (f+g) = \int_a^b f + \int_a^b g \Leftarrow$ **Sum Rule.**

Proof: Let $I_f = \int_a^b f$, $I_g = \int_a^b g$, $I_{f+g} = \int_a^b (f+g)$. Given $\varepsilon > 0 \exists \delta > 0 \ni$, for all partitions P of $[a,b]$, we have that

a. $\|P\| < \delta \Rightarrow \left|S_f^*(P) - I_f\right| < \frac{\varepsilon}{3}$

b. $\|P\| < \delta \Rightarrow \left|S_g^*(P) - I_g\right| < \frac{\varepsilon}{3}$

c. $\|P\| < \delta \Rightarrow \left|S_{f+g}^*(P) - I_{f+g}\right| < \frac{\varepsilon}{3}$

Let P be a partition of $[a,b]$ such that $\|P\| < \delta$, and choose $x_k^* \in [x_{k-1}, x_k]$, $1 \leq k \leq n$, the same for f, g and $f+g$. Note that $S_{f+g}^*(P) = S_f^*(P) + S_g^*(P)$. So, by a),b) and c), we obtain

$$\left|I_{f+g} - (I_f + I_g)\right| = \left|(I_{f+g} - S_{f+g}^*(P)) + S_{f+g}^*(P) - (I_f + I_g)\right| =$$
$$\left|(I_{f+g} - S_{f+g}^*(P)) - (I_f - S_f^*(P)) - (I_g - S_g^*(P))\right| \leq$$
$$\left|(I_{f+g} - S_{f+g}^*(P))\right| + \left|(I_f - S_f^*(P))\right| + \left|(I_g - S_g^*(P))\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

Thus, $\left|I_{f+g} - (I_f + I_g)\right| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude $I_{f+g} = I_f + I_g$ or $\int_a^b (f+g) = \int_a^b f + \int_a^b g$. ∎

7. $\int_a^b \alpha \cdot f = \alpha \cdot \int_a^b f \Leftarrow$ **Scalar Multiple Rule.**

Proof: Let $I_f = \int_a^b f$ and $I_{\alpha \cdot f} = \int_a^b \alpha \cdot f$ Given $\varepsilon > 0 \exists \delta > 0 \ni$, for all partitions P of $[a,b]$, we have

a) $\|P\| < \delta \Rightarrow \left|\alpha \cdot S_f^*(P) - \alpha \cdot I_f\right| < \frac{\varepsilon}{2}$.

b) $\|P\| < \delta \Rightarrow \left|S_{\alpha \cdot f}^*(P) - I_{\alpha \cdot f}\right| < \frac{\varepsilon}{2}$.

Let P be a partition of $[a,b]$ such that $\|P\| < \delta$. Note that $S_{\alpha \cdot f}^*(P) = \alpha \cdot S_f^*(P)$. So, by a) and b), we obtain

$$\left|\alpha\cdot I_f-I_{\alpha\cdot f}\right|=\left|\alpha\cdot I_f-\alpha\cdot S_f^*(P)+\alpha\cdot S_f^*(P)-I_{\alpha\cdot f}\right|=$$
$$\left|\left(\alpha\cdot I_f-\alpha\cdot S_f^*(P)\right)+\left(S_{\alpha\cdot f}^*(P)-I_{\alpha\cdot f}\right)\right|\leq\left|\left(\alpha\cdot I_f-\alpha\cdot S_f^*(P)\right)\right|+\left|\left(S_{\alpha\cdot f}^*(P)-I_{\alpha\cdot f}\right)\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Thus, $\left|\alpha\cdot I_f-I_{\alpha\cdot f}\right|<\varepsilon$. Since $\varepsilon>0$ was arbitrary, we conclude that $I_{\alpha\cdot f}=\alpha\cdot I_f$ or $\int_a^b \alpha\cdot f=\alpha\int_a^b f$. ■

8. $\int_a^b(\alpha\cdot f+\beta\cdot g)=\alpha\cdot\int_a^b f+\beta\cdot\int_a^b g \Leftarrow$ **Linear Rule.** This is equivalent to the Sum Rule together with the Scalar Multiple Rule.

9. $f\leq g\Rightarrow\int_a^b f\leq\int_a^b g \Leftarrow$ **Non-Decreasing Rule.**

Proof: $f\leq g\Rightarrow(g-f)\geq 0\Rightarrow\int_a^b(g-f)\geq 0$, by the Non-Negative Rule. By the Linear Rule, we then obtain

$$\int_a^b g-\int_a^b f=\int_a^b(g-f)\geq 0\Rightarrow\int_a^b f\leq\int_a^b g.$$ ■

10. $f<g\Rightarrow\int_a^b f<\int_a^b g \Leftarrow$ **Increasing Rule.**

Proof: $f<g\Rightarrow(g-f)>0\Rightarrow\int_a^b(g-f)>0$, by the Positive Rule. By the Linear Rule, we then obtain

$$\int_a^b g-\int_a^b f=\int_a^b(g-f)>0\Rightarrow\int_a^b f<\int_a^b g.$$ ■

11. $\int_a^a f\overset{\text{def}}{=}0 \Leftarrow$ **Zero Width Interval Rule.**

12. $\int_b^a f\overset{\text{def}}{=}-\int_a^b f \Leftarrow$ **Order Integration Rule.**

13. $a<c<b\Rightarrow\int_a^b f=\int_a^c f+\int_c^b f \Leftarrow$ **Additive Rule.**

Proof: Since f is continuous on $[a,b]$, it is continuous on $[a,c]$ and on $[c,b]$. Therefore, $\forall\varepsilon>0\,\exists\delta>0\ni$

a. $\forall P$ of $[a,b]$, $\|P\|<\delta \Rightarrow \left|S_f^*(P)-\int_a^b f\right|<\frac{\varepsilon}{3}$

b. $\forall Q$ of $[a,c]$, $\|Q\|<\delta \Rightarrow \left|S_f^*(Q)-\int_a^c f\right|<\frac{\varepsilon}{3}$

c. $\forall R$ of $[c,b]$, $\|R\|<\delta \Rightarrow \left|S_f^*(R)-\int_c^b f\right|<\frac{\varepsilon}{3}$

Let Q and R be a partitions of $[a,c]$and $[c,b]$, respectively, such that $\|Q\|<\delta$ and $\|R\|<\delta$. Let $P=Q\cup R$. Then $\|P\|<\delta$ and $S_f^*(P)=S_f^*(Q)+S_f^*(R)$. So,

$$\left|\int_a^b f-\left(\int_a^c f+\int_c^b f\right)\right|=\left|\int_a^b f-S_f^*(P)+S_f^*(P)-\left(\int_a^c f+\int_c^b f\right)\right|=$$

$$\left|\left(\int_a^b f-S^*(P)\right)+\left(S_f^*(Q)+S_f^*(R)\right)-\left(\int_a^c f+\int_c^b f\right)\right|=$$

$$\left|\left(\int_a^b f-S_f^*(P)\right)-\left(\int_a^c f-S_f^*(Q)\right)-\left(\int_c^b f-S_f^*(R)\right)\right|\le$$

$$\left|\left(\int_a^b f-S_f^*(P)\right)\right|+\left|\left(\int_a^c f-S_f^*(Q)\right)\right|+\left|\left(\int_c^b f-S_f^*(R)\right)\right|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$$

Thus, $\left|\int_a^b f-\left(\int_a^c f+\int_c^b f\right)\right|<\varepsilon$. Since ε was arbitrary, $\int_a^b f=\int_a^c f+\int_c^b f$. ∎

14. If f is continuous on the smallest interval that contains $a,b,$ and c, then $\int_a^b f=\int_a^c f+\int_c^b f$, no matter what the order of $a,b,$ and c are $\Leftarrow$ **Interval Additive Rule.**

Proof: Consider the case $c<b<a$. The other cases are handled in the same way. Now, $\int_a^c f+\int_c^b f=-\left(\int_c^a f\right)+\int_c^b f=-\left(\int_c^b f+\int_b^a f\right)+\int_c^b f=-\int_c^b f-\int_b^a f+\int_c^b f=-\int_b^a f=\int_a^b f$. Thus, $\int_a^b f=\int_a^c f+\int_c^b f$. The other cases are done in the same way. ∎

Exercises:

1. Let be $p(x)$ a polynomial and $a<b$. Show that $\int_a^b p(x)\,dx = \left.\int p(x)\,dx\right|_a^b$.

2. Let f be continuous on the closed interval $[a,b]$. If $F(x)=\int_a^x f$, show that $\frac{F(x+h)-F(x)}{h}=\frac{\int_x^{x+h} f}{h}$.

Fundamental Theorem of Integral Calculus (FTIC)

The relation $\int_a^b p(x)\,dx = \left.\int p(x)\,dx\right|_a^b$ in exercise 1 above is an example of the **Fundamental Theorem of Integral Calculus,** but for polynomials. We will now begin to show that this theorem also holds for any continuous f on $[a,b]$.

First Fundamental Theorem of Integral Calculus: Let f be continuous on the closed interval $[a,b]$, and let $F(x)=\int_a^x f$, $a\le x\le b$. Then

a) $F'_+(x)=f(x), x\in[a,b)$. In particular, $F'_+(a)=f(a)$.
b) $F'_-(x)=f(x), x\in(a,b]$. In particular, $F'_-(b)=f(b)$.
c) F is continuous on $[a,b]$.
d) $F'(x)=f(x), x\in(a,b)$. |

Proof:

a) Let $x\in[a,b)$ and hold fixed. Since f is continuous on $[a,b]$, , it is right-hand continuous at x. Therefore, given $\varepsilon>0\,\exists\,\delta>0\ni$
$0\le t-x<\delta\Rightarrow|f(t)-f(x)|<\varepsilon\Leftrightarrow f(x)-\varepsilon<f(t)<f(x)+\varepsilon$. Choose $0<h<\delta$. Then,

$$t\in[x,x+h]\Rightarrow 0\le t-x<h<\delta\Rightarrow f(x)-\varepsilon<f(t)<f(x)+\varepsilon\Rightarrow \int_x^{x+h}(f(x)-\varepsilon)\,dx<\int_x^{x+h}f(t)\,dt<\int_x^{x+h}(f(x)+\varepsilon)\,dx\Rightarrow$$

$$(f(x)-\varepsilon)h<\int_x^{x+h}f(t)\,dt<(f(x)+\varepsilon)h\Rightarrow f(x)-\varepsilon<\frac{\int_x^{x+h}f(t)\,dt}{h}<f(x)+\varepsilon\Rightarrow$$

$$f(x)-\varepsilon<\frac{F(x+h)-F(x)}{h}<f(x)+\varepsilon\Leftrightarrow\left|\frac{F(x+h)-F(x)}{h}-f(x)\right|<\varepsilon.$$

In summary, $0<h<\delta \Rightarrow \left|\frac{F(x+h)-F(x)}{h}-f(x)\right|<\varepsilon$. Thus, $F'_+(x)=f(x), x\in[a,b)$. In particular, $F'_+(a)=f(a)$.

b) Let $x\in(a,b]$ and hold fixed. Since f is continuous on $[a,b]$, , it is left-hand continuous at x. Therefore, given $\varepsilon>0 \exists \delta>0 \ni$
$0\le x-t<\delta \Rightarrow |f(t)-f(x)|<\varepsilon \Leftrightarrow f(x)-\varepsilon<f(t)<f(x)+\varepsilon$. Choose $0<-h<\delta$. Then,

$$t\in[x+h,x] \Rightarrow 0\le x-t<-h<\delta \Rightarrow f(x)-\varepsilon<f(t)<f(x)+\varepsilon \Rightarrow \int_{x+h}^{x}(f(x)-\varepsilon)\,dx<\int_{x+h}^{x}f(t)\,dt<\int_{x+h}^{x}(f(x)+\varepsilon)\,dx \Rightarrow$$

$$-(f(x)-\varepsilon)h<\int_{x+h}^{x}f(t)\,dt<-(f(x)+\varepsilon)h \Rightarrow f(x)-\varepsilon<\frac{\int_{x+h}^{x}f(t)\,dt}{-h}<f(x)+\varepsilon \Leftrightarrow \frac{\int_{x}^{x+h}f(t)\,dt}{h} \Rightarrow$$

$$f(x)-\varepsilon<\frac{F(x+h)-F(x)}{h}<f(x)+\varepsilon \Leftrightarrow \left|\frac{F(x+h)-F(x)}{h}-f(x)\right|<\varepsilon.$$

In summary, $0<-h<\delta \Rightarrow \left|\frac{F(x+h)-F(x)}{h}-f(x)\right|<\varepsilon$. Thus, $F'_-(x)=f(x), x\in(a,b]$. In particular, $F'_-(b)=f(b)$.

c) Since differentiability implies continuity, items a) and b) imply that F is continuous on $[a,b]$.

d) Also, from items a) and b), we see that $x\in(a,b) \Rightarrow F'_+(x)=f(x)=F'_-(x)$. Thus, $F'(x)=f(x), x\in(a,b)$. ∎

Second Fundamental Theorem of Integral Calculus: Let f be continuous on the closed interval $[a,b]$, and let $F(x)=\int_a^x f$, $a\le x\le b$. If G is an anti-derivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx=G(b)-G(a).$$

Proof: By the First FTIC, $F'(x)=f(x)$ on (a,b), and since $G'(x)=f(x)$ on (a,b), we have $F'(x)=G'(x)$ on (a,b). Finally, since F,G are continuous on $[a,b]$, we have $F'(x)=G(x)+C$ on $[a,b]$, for some constant C. What is C? Now,
$F(a)=\int_a^a f(x)\,dx=0 \Rightarrow 0=F(a)=G(a)+C \Rightarrow C=-G(a)$. So, $\int_a^x f=F(x)=G(x)-G(a)$ on $[a,b]$, In particular,

$$\int_a^b f=F(b)=G(b)-G(a).$$ ∎

Stated another way, the Second FTIC says

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Big|_a^b$$

Properties of the Riemann Integral (Continued)

Let f, g, and g' be continuous on $[a,b]$ and $u(x)$, $v(x)$ appropriately differentiable. Then,

15. $\dfrac{d}{dx}\displaystyle\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))\dfrac{dv}{dx} - f(u(x))\dfrac{du}{dx}$. $\Leftarrow$ **Leibniz Integral Rule.**

    Proof: By the First FTIC and the Chain Rule, we have

    $\dfrac{d}{dx}\displaystyle\int_{u(x)}^{v(x)} f(t)\,dt = \dfrac{d}{dx}\int_{u(x)}^{c} f(t)\,dt + \dfrac{d}{dx}\int_{c}^{v(x)} f(t)\,dt = \dfrac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = \dfrac{d}{dx}\int_{c}^{v(x)} f(t)\,dt - \dfrac{d}{dx}\int_{c}^{u(x)} f(t)\,dt =$

    $f(v(x))\dfrac{dv}{dx} - f(u(x))\dfrac{du}{dx}$. ∎

16. Let $f \geq 0$. Then $\displaystyle\int_a^b f(t)\,dt = 0 \Leftrightarrow f = 0$ on $[a,b]$. $\Leftarrow$ **Zero Rule.**

    Proof: The "if part" is trivial. Therefore, we shall only prove the "only if" part. So, assume that $\displaystyle\int_a^b f(t)\,dt = 0$. Since

    $f \geq 0$, $\displaystyle\int_a^x f(t)\,dt = 0, \forall x \in [a,b]$. Now, $\displaystyle\int_a^x f(t)\,dt = 0, \forall x \in [a,b] \Rightarrow D_x \int_a^x f(t)\,dt = 0, \forall x \in (a,b) \Rightarrow$
    $f(x) = 0$, $\forall x \in (a,b) \Rightarrow f(x) = 0$, $\forall x \in [a,b]$, by continuity. ∎

17. $\displaystyle\int f(g(x))g'(x)\,dx = \int f(u)\,du \Big|_{u=g(x)}$ $\Leftarrow$ ***U*-Substitution Rule.**

    Proof: Let $D_u(F(u)) = f(u)$. Then, $D_x F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$. Thus,

    $\displaystyle\int f(g(x))g'(x)\,dx = F(g(x)) + C = F(u) + C \Big|_{u=g(x)} = \int f(u)\,du \Big|_{u=g(x)}$. ∎

18. $\displaystyle\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ $\Leftarrow$ **Change of Variable Rule.**

    Proof: Let $D_u(F(u)) = f(u)$. Then, $\displaystyle\int_a^b f(g(x))g'(x)\,dx = F(g(x)) \Big|_a^b = F(u) \Big|_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u)\,du$. ∎

19. $\int_{-a}^{a} f = 2\int_{0}^{a} f$, where $f(-x) = f(x)$. $\Leftarrow$ Here f is said to be **even**. A Symmetry Law.

20. $\int_{-a}^{a} f = 0$, where $f(-x) = -f(x)$. $\Leftarrow$ Here f is said to be **odd**. A Symmetry Law.

21. $\left|\int_a^b f\right| \le \int_a^b |f| \Leftarrow$ **Absolute Value Rule.**

Proof: Clearly, we have $-|f| \le f \le |f|$ on $[a,b]$. By the Comparison Rule, we then have $[a,b]$. By the Comparison Rule, we then have $-\int_a^b |f| \le f \le \int_a^b |f| \Rightarrow \left|\int_a^b f\right| \le \int_a^b |f|$. ∎

22. $f_{av} \overset{\text{def}}{=} \dfrac{\int_a^b f}{b-a} \Leftarrow$ Definition of the **average value** of f over $[a,b]$.

23. $f(c) = \dfrac{\int_a^b f}{b-a}$, $c \in (a,b) \Leftarrow$ **First Mean Value Theorem for Integrals** (First MVTI).

Proof: The theorem is trivial if f is constant on $[a,b]$. Therefore, we may assume that f is not constant on $[a,b]$. By the Max-Min Rule, we have that $m(b-a) \le \int_a^b f \le M(b-a)$. If $m(b-a) = \int_a^b f$, then $\int_a^x (f(t)-m)\,dt = 0, \forall x \in [a,b]$. This would then imply that $f = m$ on $[a,b]$. This contradicts the fact that f is not constant on $[a,b]$. Thus, $m(b-a) < \int_a^b f$. In the same way, we also have $\int_a^b f < M(b-a)$. Hence, $m < \dfrac{\int_a^b f}{b-a} < M$. Since f is continuous on $[a,b]$, by the Intermediate Value Theorem, there exists $c \in (a,b)$ such that $f(c) = \dfrac{\int_a^b f}{b-a}$. ∎

24. $\int_{a+p}^{b+p} f = \int_a^b f$, where f is periodic with period $p \Leftarrow$ **Periodic Rule for Integrals.**

Proof: $\int_{a+p}^{b+p} f(x)\,dx = \int_{a+p}^{b+p} f(x+p)\,dx = \int_a^b f(u)\,du \Leftarrow$, where we have used the change of variable $u = x + p$. ∎

Applications of the Second FTIC

The Second FTIC can be restated in this way:

$$\int_a^x F'(s)\,ds = F(s)\Big|_a^x = F(x) - F(a)$$

A function such as F is called an accumulation function because it accumulates area under the graph of its derivatives for $x \geq a$.

Position as an Accumulation Function:

Exercise 1: An object moves along a coordinate line with velocity $v(t) = 6t^2 - 6$ units per second. Its initial position at time $t = 0$ is 2 units to the left of the origin.

a) Find the position of the object 3 seconds later.

b) Find the total distance traveled by the object during those 3 seconds.

Exercise 2: An object moves along a coordinate line with velocity $v(t) = \sin 2t$ units per second. Its initial position at time $t = 0$ is 1 unit to the right of the origin.

a) Find the position of the object π seconds later.

b) Find the total distance traveled by the object during those π seconds.

Some Quantity as an Accumulation Function:

If $F(t)$ measures the amount of some quantity at time t, then the Second FTIC says that the accumulated rate of change from time $t = a$ to $t = b$ is equal to the net change in that quantity over the interval $[a,b]$.

Exercise 3: Water leaks out of a 55-gallon tank at the rate $V'(t) = 11 - 1.1t$, where t is measured in hours and V in gallons. Initially, the tank is full.

a) How much water leaks out of the tank between $t = 3$ and $t = 5$ hours?

b) How long does it take until there are just 5 gallons remaining in the tank?

Exercise 4: Water leaks out of a 200-gallon tank at the rate $V'(t) = 20 - t$, where t is measured in hours and V in gallons. Initially, the tank is full.

a) How much water leaks out of the tank between $t = 10$ and $t = 20$ hours?

b) How long does it take until the tank is drained completely?

Finding Area Between Two Curves

Slicing, Approximating and Integrating with Respect to the *X-Axis*

Consider two curves defined by the two functions $y = f(x), y = g(x)$ continuous on the closed interval $[a,b]$. We want to compute the area of the region Ω between these two curves from $x = a$ to $x = b$ (see Figure1).

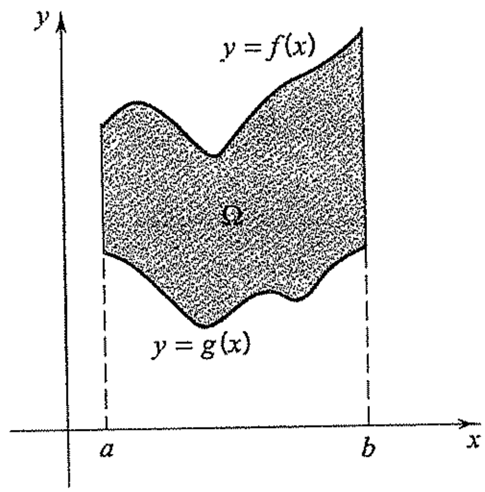


Figure1

Let $P = \{x_0 = a, x_1, \ldots, x_{i-1}, x_i, \ldots, x_{n-1}, x_n = b\}$ be a partition of $[a,b]$. In this way, we **slice** the region Ω into $n \in \mathbb{N}$ subregions. Let $\Delta A_i, 1 \le i \le n,$ denote the the area of the i^{th} subregion. We then **approximate** ΔA_i by the rectangular area $(f(x_i^*) - g(x_i^*))\Delta x_i$, where x_i^* is any sample point in $[x_{i-1}, x_i]$. Thus, $\Delta A_i \approx (f(x_i^*) - g(x_i^*))\Delta x_i$, $1 \le i \le n$. Intuitively, we feel that as $\|P\| \to 0$, these approximations become better. Therefore, we define A, area of Ω, by integrating $(f - g)$ over $[a,b]$. That is,

$$A \overset{def}{=} \lim_{\|P\| \to 0} \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\Delta x_i \equiv \int_a^b (f(x) - g(x))\,dx.$$

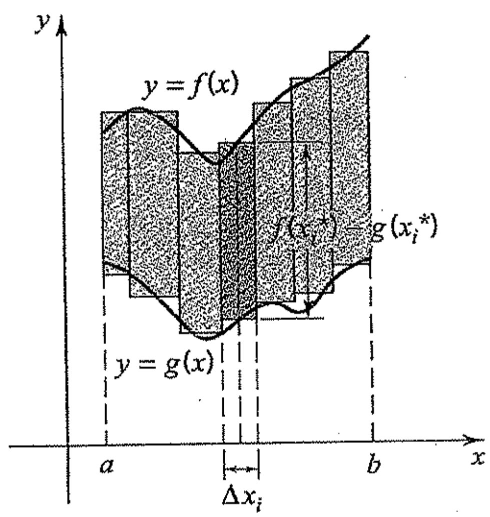


Figure 2

If we are given two curves defined by the two functions $x = F(y), x = G(y)$ continuous on the closed interval $[c, d]$, then we employ the same process as above only with respect to the $y-axis$, as shown in Figure 3 and Figure 4 below.

Slicing, Approximating and Integrating with Respect to the *Y-Axis*

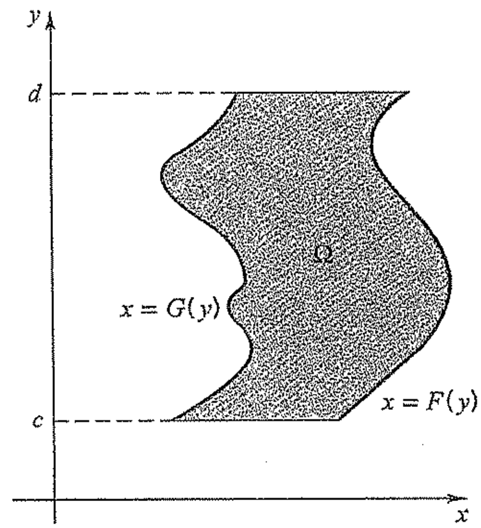


Figure 3

In this case, we define

$$A \overset{def}{=} \lim_{\|P\| \to 0} \sum_{i=1}^{n} \left(F(y_i^*) - G(y_i^*)\right) \Delta y_i \equiv \int_c^d (F(y) - G(y))\,dy$$

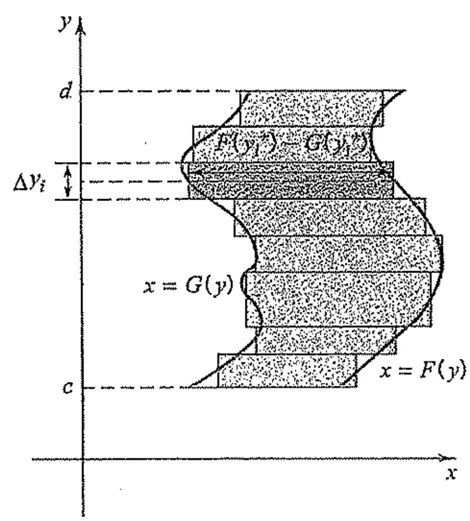
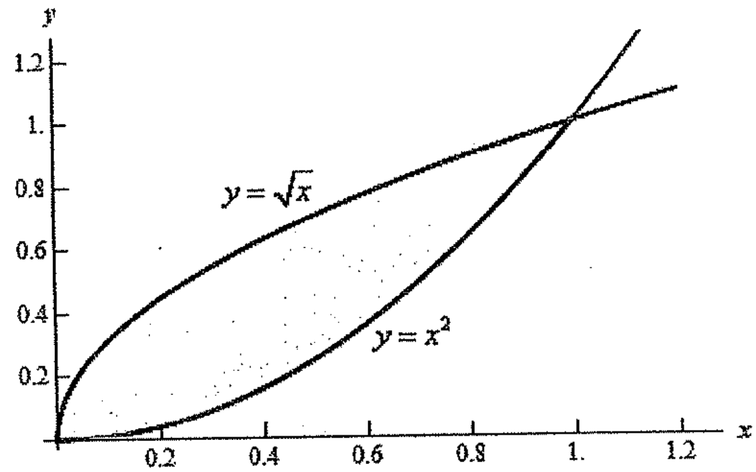


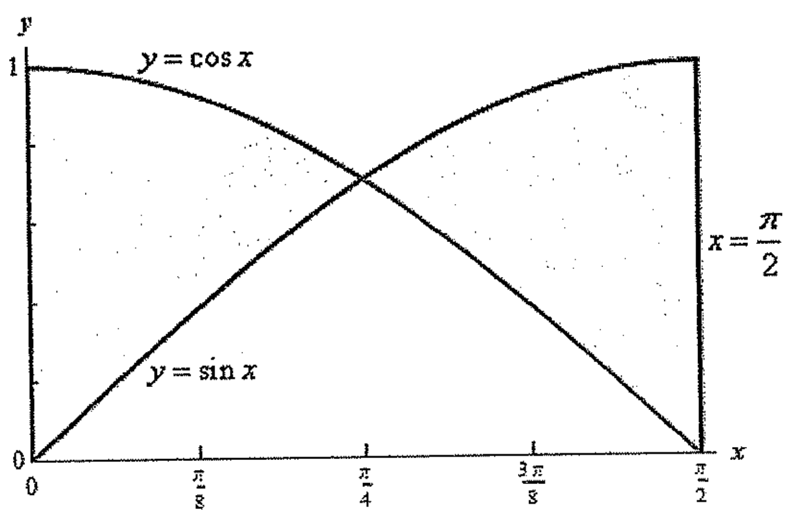
Figure 4

:xercise 1: Determine the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.

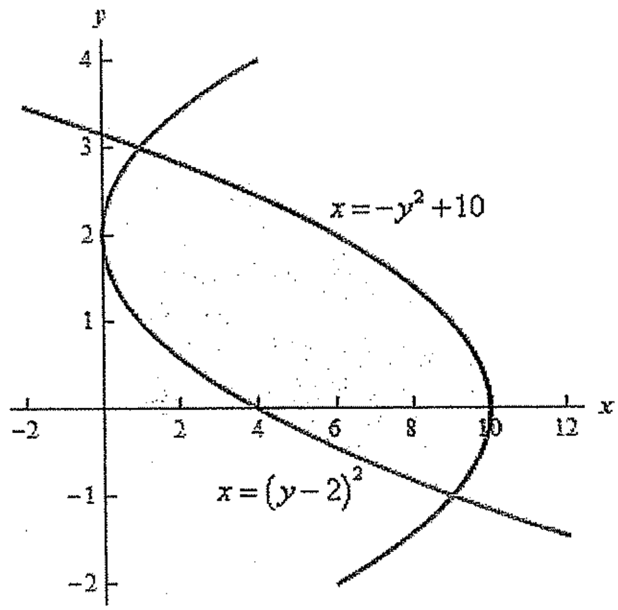
'irst of all, just what do we mean by "area enclosed by". This means that the region we're interested in must have one of the two urves on every boundary of the region. So, here is a graph of the two functions with the enclosed region shaded.



Exercise 2: Determine the area of the region enclosed $y = \sin x, y = \cos x, x = \frac{\pi}{2}$ and the $y-$axis. Here is a sketch of the region.



Exercise 3: Determine the area of the region bounded $x = -y^2 + 10$ and $x = (y-2)^2$. Here is a sketch of the region.

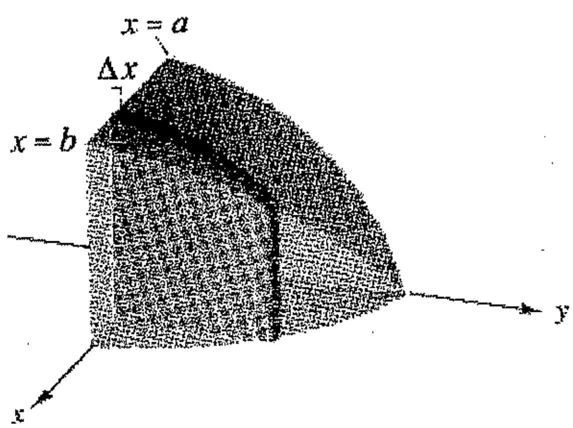


Determining Volume

Volumes by Cross Sections

A cross section of a solid is a plane figure obtained by the intersection of that solid with a plane. The cross section of an object therefore represents an infinitesimal "slice" of a solid, and may be different depending on the orientation of the slicing plane. While the cross section of a sphere is always a disk, the cross section of a cube may be a square, hexagon, or other shape. Some other common cross sections are rectangles, triangles, semicircles, and trapezoids.

Integration allows us to calculate the volumes of such solids. That is, we may define the volume of a solid as a limit of a Riemann sum of cross sectional areas $A(x)$. This is similar to the way we defined the area between two curves. Let S be a solid that lies between $x=a$ and $x=b$. Let the continuous function $A(x)$ represent the cross-sectional area of S in the plane through the point x and perpendicular to the x-axis, as seen in Figure (a) below. The volume of S is then given by Formula (a) below.

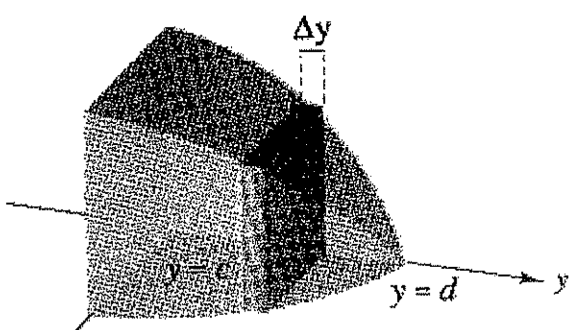


(a) Cross sections perpendicular to x-axis

$$V = \int_a^b A(x)\,dx = \lim_{\|P\|\to 0} S^*_{A(x)}(P)$$

Figure (a) Formula (a)

On the other hand, let S be a solid that lies between $y=c$ and $y=d$. Let the continuous function $A(y)$ represent the cross-sectional area of S in the plane through the point y and perpendicular to the y-axis as seen in Figure (b) below. The volume of S is then given by Formula (b) below.

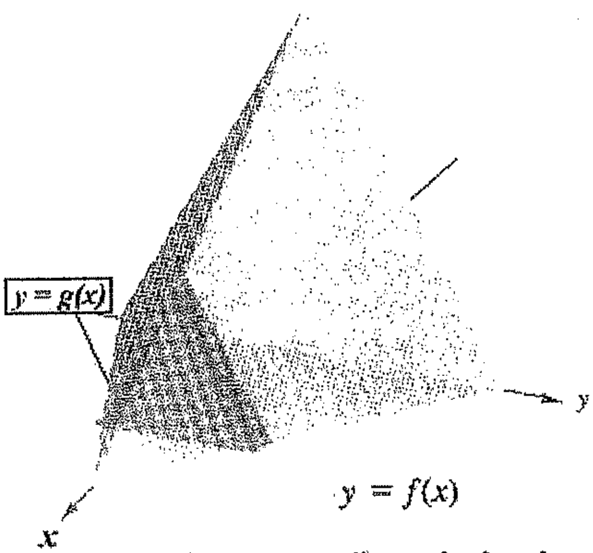


(b) Cross sections perpendicular to y-axis

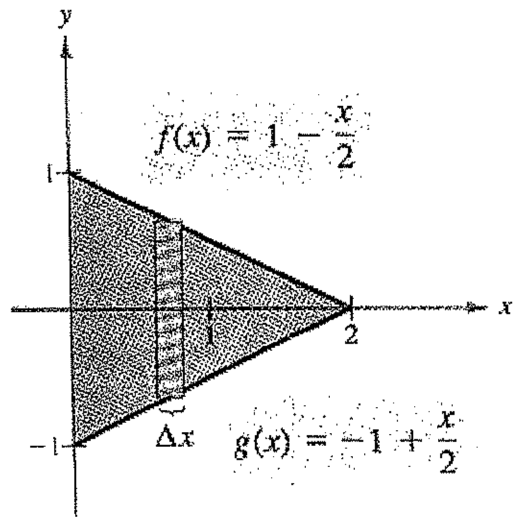
$$V = \int_c^d A(y)\,dy = \lim_{\|P\|\to 0} S^*_{A(y)}(P)$$

Figure (b) Formula (b)

Example: Find the volume of the solid shown below. The base of the solid is the region bounded by the lines $f(x)=1-\frac{x}{2}$, $g(x)=-1+\frac{x}{2}$, and $x=0$. The cross sections perpendicular to the x-axis are equilateral triangles.



Cross sections are equilateral triangles.



Triangular base in xy-plane

Solution: The base of each equilateral triangular cross section, area of each equilateral triangular cross section and corresponding volume element of the solid are

Base of Triangular Cross Section: $\text{Base}=\left(1-\frac{x}{2}\right)-\left(-1+\frac{x}{2}\right)=2-x$

Cross Sectional Area of Equilateral Triangle: $A(x)=\frac{\sqrt{3}}{4}(\text{Base})^2=\frac{\sqrt{3}}{4}(2-x)^2$

Volume Element: $dV=A(x)dx=\frac{\sqrt{3}}{4}(2-x)^2\,dx$

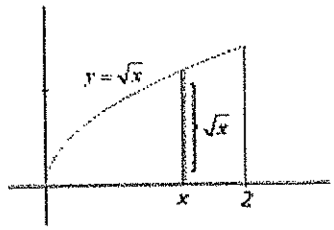
Because x ranges from 0 to 2, the volume of the solid is

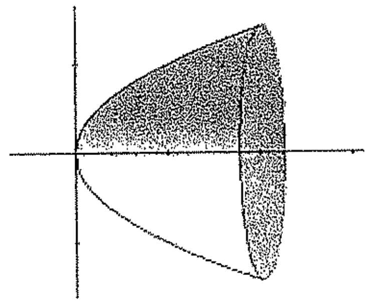
$$V=\int_0^2 dv=\int_0^2\frac{\sqrt{3}}{4}(2-x)^2\,dx=\frac{\sqrt{3}}{4}\int_0^2(x-2)^2\,dx=\frac{\sqrt{3}}{4}\frac{(x-2)^3}{3}\bigg|_0^2=\frac{2\sqrt{3}}{3}$$

Now that we have the definition of volume, the challenging part is to find the function of the area of a given cross section. This process is quite similar to finding the area between curves.

Solids of Revolution

Most volume problems that we will encounter will require us to calculate the volume of a **solid of revolution**. These are solids that are obtained when a plane region is rotated about some line. A typical volume problem would ask, "Find the volume of the solid generated by rotating the region bounded by the some curve(s) about some specified line." Since the region is rotated about a specific line, the solid obtained by this rotation will have a disk-shaped cross-section. We know from simple geometry that the area of a circle is given by $A=\pi r^2$. For each cross-sectional disk, the radius is determined by the curves that bound the region. If we sketch the region bounded by the given curves, we can easily find a function to determine the radius of the cross-sectional disk at point x.





For example, the figures above illustrates this concept. The figure to the left shows the region bounded by the curve $y=\sqrt{x}$ and the x-axis and the lines $x=0$ and $x=2$. The figure in the center shows the 3-dimensional solid that is formed when the region from the first figure is rotated about the x-axis. The figure to the right shows a typical cross-sectional disk. A disk for a given value x between 0 and 2 will have a radius of $y=\sqrt{x}$. The area of the disk is given by $A=\pi\left(\sqrt{x}\right)^2$ or equivalently, $A=\pi x$. Once we find the area function, we simply integrate from a to b to find the volume. In this example, the volume V in question is given by

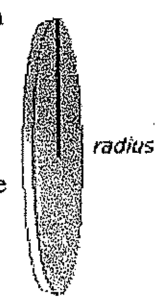
$$V=\int_a^b A(x)\,dx=\int_0^2 \pi x\,dx=\pi\int_0^2 x\,dx=\pi\frac{x^2}{2}\Bigg|_0^2=2\pi.$$

Variations of Volume Problems

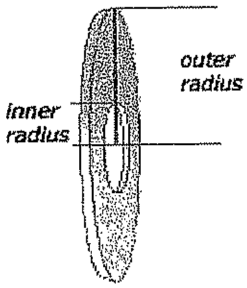
There are two variations in problems of solids of revolution that we will consider. The first factor that can vary in this type of volume problem is the axis of rotation. What if the region from the figure above was rotated about the y-axis rather than the x-axis? We would end up with a different function for the radius of the cross-sectional disk. The function would be written with respect to y rather than x, so we would have to integrate with respect to y. In general, we can use the following rule.

If the region bounded by the curves and the lines $x=a$ and $x=b$ is rotated about an axis parallel to the x-axis, write the integral with respect to x. If the axis of rotation is parallel to the y-axis, write the integral with respect to y.

The second factor that can vary in this type of volume problem is whether or not the axis of rotation is part of the region that is being rotated. In the first case, each cross section that is generated will be a disk while in the second case, each cross section that is generated will be washer shaped. This creates two separate styles of problems:



The Disk Method: The disk method is used when the cross sections are disk shaped. The radius of a cross section is determined by a single function, $f(x)$. The area of the disk is given by the formula, $A=\pi r^2 \Rightarrow A(x)=\pi f^2(x)$, where $r=f(x)$. The corresponding volume would then be $V=\int_a^b A(x)\,dx=\int_a^b \pi f^2(x)\,dx$. The figure to the right shows a typical cross-sectional disk.



The Washer Method: The washer method is used when the cross sections are washer shaped. The radius of a cross section is determined by a two functions, $f(x)$ and $g(x)$. This gives us two separate radii, an outer radius, from $f(x)$ to the axis of rotation and an inner radius from $g(x)$ to the axis of rotation. The area of the washer is given by the formula, $A(x)=\pi\left[f^2(x)-g^2(x)\right]$, where outer radius $=f(x)$ and inner radius $=g(x)$. The corresponding volume would then be $V=\int_a^b A(x)\,dx=\int_a^b \pi\left[f^2(x)-g^2(x)\right]dx$. The figure to the left shows a typical cross-sectional washer.

Exercises:

1. Find the volume of the solid that is generated by rotating the region bounded by the curves $y=x^2, x=0$, and $x=2$ about the x-axis.

2. Find the volume of the solid that is generated by rotating the region bounded by the curves $y=x^2, x=0$, and $x=2$ about the y-axis.

3. Find the volume of the solid that is generated by rotating the region bounded by the curves $y=\sqrt{x}$ and $y=x$ about t (a) the x-axis and about (b) the y-axis.

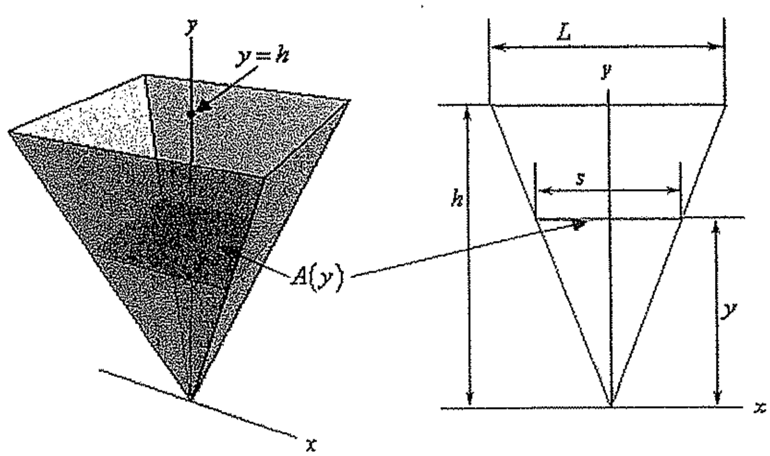
4. Find the volume of the solid that is generated by rotating the region bounded by the curves $y=\sqrt{x}, x+y=6$, and $y=1$ about the x-axis.

5. Find the volume of a sphere of radius r.

6. Find the volume of a circular cone of radius r and height h.

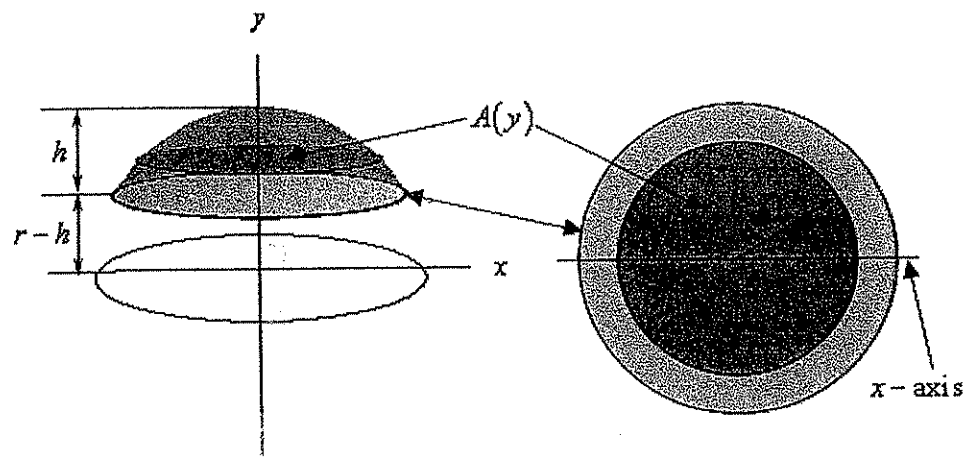
7. The base of a solid is the region between the parabolas $x=y^2$ and $x=3-2y^2$. Find the volume of the solid given that the cross sections perpendicular to the x-axis are squares.

8. Find the volume of a pyramid whose base is a square with sides of length L and whose height is h (see the figure below).

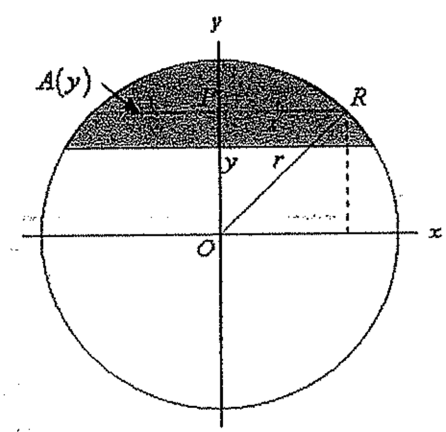


Three and Two Dimensional Views of a Cross Section

9. For a sphere of radius r find the volume of the cap of height h (see the figures below).

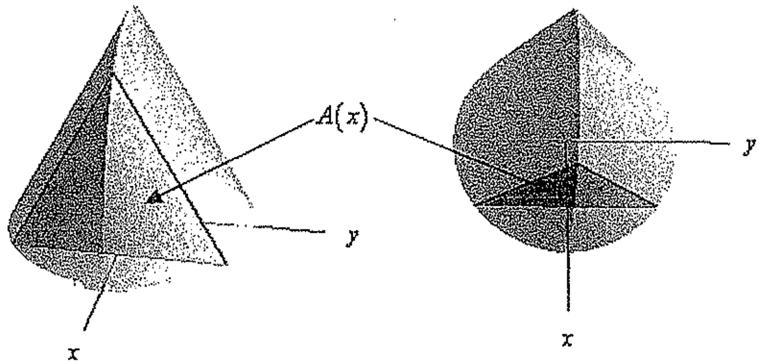


Three and Two Dimensional Views of a Cross Section

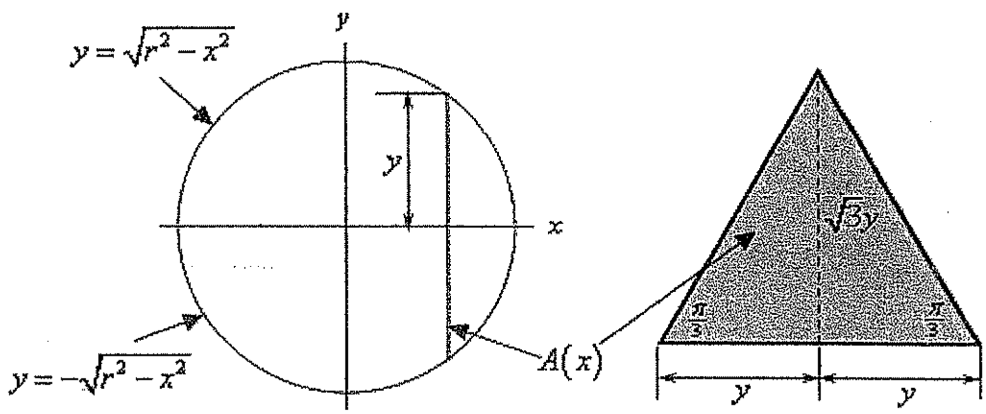


Two Dimensional View of a Cross Section

10. Find the volume of the solid whose base is a disk of radius r and whose cross-sections are equilateral triangles (see the figures below).



Three Dimensional Views of a Cross Section



Two Dimensional Views of a Cross Section

The axes of two equal cylinders of radius r intersect perpendicularly. Find the volume common to the two cylinders.

Hint: First, the equations of the two cylinders are given by $x^2 + y^2 = r^2$ and $x^2 + z^2 = r^2$. In the figure below you see one eighth of the total volume in the first quadrant with a typical cross sectional area shaded. In fact, the cross sections are squares. To see this, from the figure we first note that the height of the rectangle is z and the width is y. From the equations, we then obtain $y^2 = r^2 - x^2 = z^2 \Rightarrow y = z$. Thus, our cross sections are squares.