Math 3023 Exam II 10/22/2013

To: Professors of Math 3023

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The students are Responsible for the Following Topics and Procedures

1) **Lines**
   a) The student must be able to determine a line $l(x)$ given various conditions:
      i. Two points of $l(x)$
      ii. Slope and a point of $l(x)$
      iii. Slope and either the x –intercept or y –intercept of $l(x)$
      iv. Parallel line $l_{||}(x)$ and a point of $l(x)$
      v. Perpendicular line $l_{\perp}(x)$ and a point of $l(x)$
   b) The student must be familiar with the general form of a line

$$Ax + By = C, A \neq 0 \vee B \neq 0$$

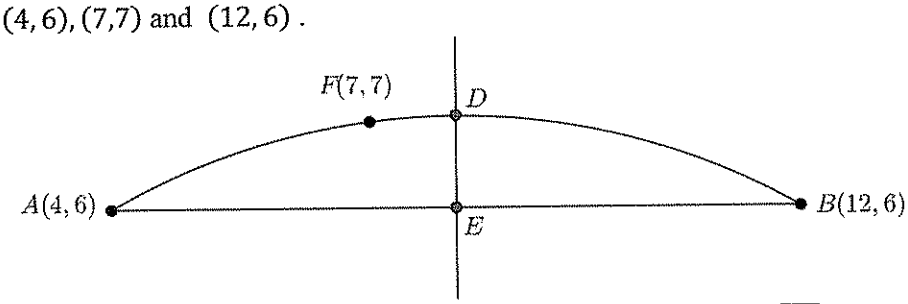
2) **Circles**
   a) The student must know the definition of a circle as a locus of points
   b) The student must know the general equation of a circle

$$x^2 + y^2 + Ax + By + C = 0$$

   c) The student must know the standard form of a circle so as to determine its center and radius.

$$(x-h)^2 + (y-k)^2 = r^2$$

   d) The student must be able to place b) in standard form c) by completing the square, for given values of A and B . Exercises: Place in standard form:
      i. $x^2 + y^2 - 3x - 9y + 25 = 0$
      ii. $x^2 + y^2 + y + 3 = 0$
   e) The student must be able to determine the equation of a circle in standard form given three points, in which one of the points is $(0,0)$ or two of the points have the same ordinate. Example of the latter: Determine the circle passing through the points $(4,6), (7,7)$ and $(12,6)$.



The center must lie on the perpendicular bisector of the segment $\overline{AB}$. Therefore, the center is $C(8,k)$, k to be determined as follows. We have then

$$r = \text{dist}(A,C) = \text{dist}(F,C)$$

Hence, $= \sqrt{(4-8)^2 + (6-k)^2} = \sqrt{(7-8)^2 + (7-k)^2}$. Solving, we get $k = -1$. Finally, the center is $C(8,-1)$ and the radius $r = \sqrt{65}$. Therefore, the equation of the circle is

$$(x-8)^2 + \left(y-(-1)\right)^2 = 65$$

Exercises: Determine the circle that passes through the following points:

i. (3, 2), (0, 0), and (5, 3)
ii. (-1, 2), (-1, 6), and (-9, 6)

f) The student must understand the relationship between circles and their tangent lines under various conditions:

i. Determine the tangent line to a given circle through a given point on the circle
ii. Determine the circle whose center is on the line $y = 2x$ and is tangent to the $x-$axis at (3, 0)

3) **Parabolas**

a) The student must know the definition of a parabola as a locus of points

b) The student must know the standard forms of a parabola and corresponding orientation of its graph.

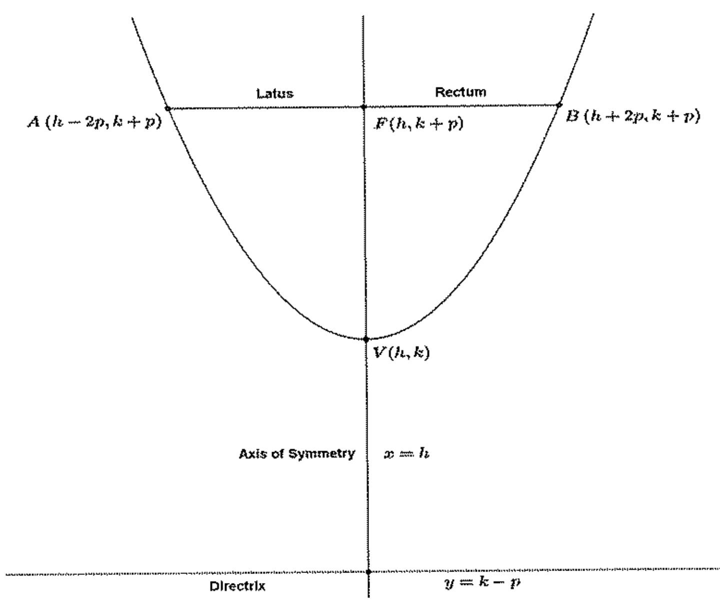
i. $(y-k) = \frac{(x-h)^2}{4p}$, opens ↑
ii. $(y-k) = -\frac{(x-h)^2}{4p}$, opens ↓
iii. $(x-h) = \frac{(y-k)^2}{4p}$, opens →
iv. $(x-h) = -\frac{(y-k)^2}{4p}$, opens ←

c) The student must know how to complete the square in order to realize the standard form of a parabola

d) The student must know how to extract the parameters from the standard form of a parabola in order to make an accurate sketch of the parabola. For example, if given

$$(y-k) = \frac{(x-h)^2}{4p}$$

then



e) Applications of the Parabola

1. Finding Maxima and Minima
    a. Find the dimensions of the rectangle that is inscribed in a right triangle with legs of length 3 and 5 and has the greatest area. Also find that area
    b. Find two real numbers whose difference is 36 and whose product is a minimum.
2. Finding Tangent Lines.
    a. Example: If $y = f(x) = x^2$, find the tangent line to $f(x)$ at the point $P(x_0, x_0{}^2)$. Consider our parabola in two different situations as shown in Figure 1 and Figure 2 below

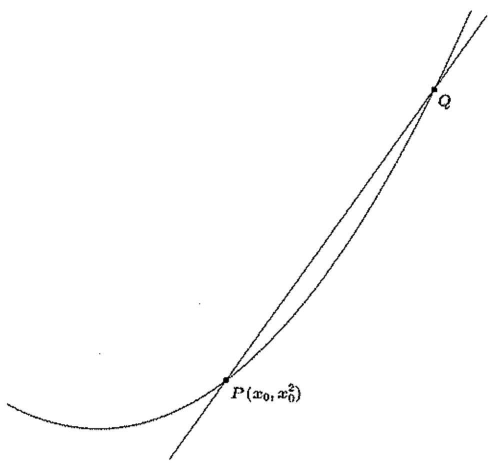


Figure 1

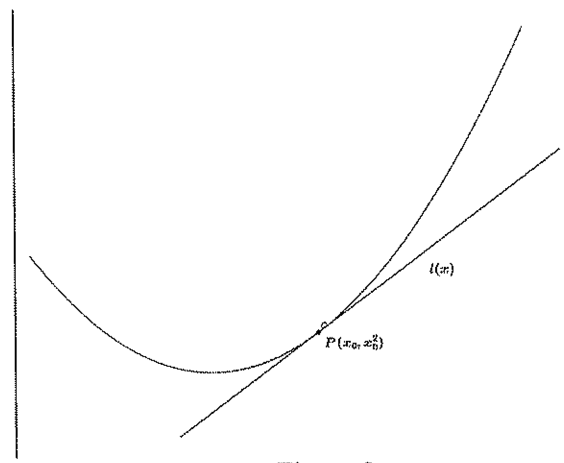


Figure 2

Basic Idea: In Figure 1, a non-tangent line through *P* will intersect our parabola in another point *Q*, while, in Figure 2, the tangent line through *P* only intersects our parabola in *P*. This means that if we use the quadratic equation to solve for the points of intersection of $l(x)$ and $f(x)$, the discriminant must be zero. This will determine the slope of the tangent line we seek. Thus,

$$f(x) = l(x)$$

$$x^2 = m(x - x_0) + x_0{}^2$$

$$x^2 - mx + (mx_0 - x_0{}^2) = 0$$

$$x_0 = x = \frac{-(-m) \pm \sqrt{D}}{2} = \frac{m}{2} \Rightarrow m = 2x_0$$

Our tangent line is then

$$l(x) = 2x_0(x - x_0) + {x_0}^2.$$

b. Exercises. Find the tangent lines to the following parabolas through the indicated points.
   i. $y = x^2 + 2x$ at (1,3)
   ii. $y = 3x^2 - x$ at (2,10)

4) **Functions**
   a) The student must know the definition of a function
   b) The student must know how to determine the domain and range of a function
   c) The student must know how to work with piecewise defined functions
   d) The student must know how to graph functions such as linear, quadratic and their piecewise combinations.

5) **Combing Functions**
   a) The student must know how to work with the algebra of functions:
      i. Sum and Difference: $f \pm g$
      ii. Scalar Multiple: af
      iii. Product: $f \cdot g$
      iv. Reciprocal: $\frac{1}{g}$
      v. Quotient: $\frac{f}{g}$
      vi. Power: $(f)^n$
      vii. Root: $\sqrt[n]{f}$
   b) The student must know how to work with the composition of functions:
      i. $f \circ g$
      ii. $g \circ f$
      iii. Exercises. If $f(x) = \sqrt{x+5}$ and $g(x) = (x-2)^2 - 7$, determine
         1. $f \circ g$ and its domain
         2. $g \circ f$ and its domain

6) **Average Rate of Change**
   a) The student must know how to compute and simplify $\frac{f(x+h)-f(x)}{h}$
   b) The student must know how to interpret $\frac{f(x+h)-f(x)}{h}$ in verbal problems

7) **Transformations of Functions**
   a) The student must know the basic transformations:
      i. Scale Change: $S_s(x) = s \cdot x$, includes both horizontal and vertical
      ii. Translation: $T_t(x) = x + t$, includes both horizontal and vertical
      iii. Reflection: $R(x) = -x$, includes both horizontal and vertical
   b) The student must know how to express the transformation of a function f as a composition using the basic transformations. Example: Express $g(x) = 3f(2x)$ as a composition of f and the basic transformations: $g(x) = (S_3 \circ f \circ S_2)(x)$
   c) Exercises: Express as a composition:
      i. $g(x) = -f(x-3)$
      ii. $g(x) = f(-x) - 3$

iii. $g(x) = 3f(-2x + 5) + 7$

d) The student must be able to sketch the transformation of a given function in the same system of coordinates as the original function

8) **Testing for Symmetry**

a) The student must know how to determine if a given function f is symmetric with respect to the $y -$ axis by applying the test $f(-x) = f(x)$

b) The student must know how to determine if a given function f is symmetric with respect to the origin by applying the test $f(-x) = -f(x)$