Quantification Theory

Let U be a universal set and $P(x)$ an open sentence on U. In science, we are always making assertions of the form

(a) "There exists an x in U such that $P(x)$ holds"

(b) "For all x in U, $P(x)$ holds"

Statement (a) is symbolized by

$$(\exists x \in U)P(x) \text{ or } (\exists x)(x \in U \wedge P(x))$$

This is an existentially quantified statement and $(\exists x)$ is the existential quantifier.

Statement (b) is symbolized by

$$(\forall x \in U)P(x) \text{ or } (\forall x)(x \in U \rightarrow P(x))$$

This is a universally quantified statement and $(\forall x)$ is the universal quantifier.

Exercises: Formalize the following statements using quantifier logic

1. Some integer is larger than 23
2. A positive integer is not negative.
3. No natural number is less than 0.
4. No positive integer is les than 1.
5. No prime is smaller than 2.
6. The product of two positive integers is positive.
7. The product of two negative integers is positive.
8. The product of a positive integer and negative integer is negative.
9. The sum of two even integers is even.
10. Every even integer is twice some integer.
11. Every odd integer is one more than twice some integer.
12. The square of an even integer is even.
13. The square root of an even squared integer is even.
14. The square of an odd integer is odd.
15. The square root of an odd squared integer is odd.
16. The square root of a positive real number less than 1 is larger than the number.
17. The square root of a positive real number larger than 1 is smaller than the number.
18. The square root of a positive real number is positive.
19. The square root of a negative real number is not a real number.
20. The square root of a negative real number is the product of i and the square root of the absolute value of the number.

Exercises: Formalize the following statements using quantifier logic.

1. For every non zero real number there exists a nonzero real number such that the product is 1.
2. There exists a non zero real number for every non zero real number such that the product is 1.
3. Between every pair of distinct rational numbers there is some irrational.

Exercise Set 3: Determine the validity of the following arguments.

(1)

If Spiderman is a superhero, then Spiderman battles evil.
Spiderman is a superhero.

$\therefore$ *Spiderman battles evil.*

Symbolize the argument. Let p symbolize *Spiderman is a superhero* and q symbolize *Spiderman battles evil,* everywhere in our argument. Then the argument form of this argument is exhibited in Figure 16a.

$$\begin{array}{l} p \Rightarrow q \\ \underline{p \qquad\quad} \\ \therefore\ q \end{array}$$

Construct an implication in which the hypothesis is the conjunction of the premises and the consequent is the argument conclusion.

$$\big(\big((p \Rightarrow q) \wedge p\big) \Rightarrow q\big)$$

Construct a truth table of this implication to determine if the last column consists of only T's.

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge p$	$\big(\big((p \Rightarrow q) \wedge p\big) \Rightarrow q\big)$
T	T	T	T	T
T	$\perp$	$\perp$	$\perp$	T
$\perp$	T	T	$\perp$	T
$\perp$	$\perp$	T	$\perp$	T

(2)

If all fruit are seedless, then some fruit is seedless.
All fruit are seedless.

$\therefore$ *Some fruit is seedless.*

(3)

If Garfield is a dog, then Garfield is a cat.
Garfield is cat.

$\therefore$ *Garfield is a dog.*

(4) A more difficult argument. Symbolize each simple statement using the first letter of each underlined word as a mnemonic for the simple statement that contains that word.

Either logic is difficult or not many students like it. If mathematics is easy, then logic is not difficult. Therefore, if many students like logic, then mathematics is not easy.

(5) *This baby is illogical. If this baby can manage a crocodile, then it is not despised. If this baby is illogical, then it is despised. Therefore, this baby cannot manage a crocodile.*

(6) *If there are no government subsidies of agriculture, then there are government controls of agriculture. If there are government controls of agriculture, there is no agricultural depression. There is either an agricultural depression or overproduction. As a matter of fact, there is no overproduction. Therefore, there are government subsidies of agriculture.*

4. Between every pair of distinct irrational numbers there is some rational.
5. Between ever pair of distinct real numbers there is a rational and an irrational.
6. You can fool some of the people all of the time.
7. You can fool all the people some of the time.
8. You can't fool all the people all the time.
9. You can't fool a person all the time.
10. Every even positive integer greater than two can be written as the sum of two primes.
11. Everybody likes somebody.
12. Somebody likes somebody.
13. Everybody likes everybody.
14. Somebody likes everybody.
15. Nobody likes everybody.
16. Somebody likes nobody.
17. There are exactly two purple mushrooms.
18. I shave all those and only those who do not shave themselves.

Rules of Quantifier Negation:

$$\neg(\forall x \in U)P(x) \equiv (\exists x \in U)(\neg P(x))$$

$$\neg(\exists x \in U)P(x) \equiv (\forall x \in U)(\neg P(x))$$

Exercises:

1. Use quantifiers to determine which of the following is logically equivalent to the negation of the statement "All snakes are poisonous"? What is the universal set?
    a. All snakes are not poisonous.
    b. Some snakes are poisonous.
    c. Some snakes are not poisonous.
2. Find the statements that are logically equivalent to the negation of each of the following statements.
    a. All snakes are reptiles.
    b. Some horses are gentle.
    c. All female students are either attractive or smart.
    d. No baby is not cute.
    e. No elephant can fly.