Symbolic Logic and Reasoning (V7)

We all use arguments in our daily life. We use them in philosophy, in the sciences, in the humanities and every day in life with family and friends. Arguments are the tools we use in reasoning to convince someone of a point of view. And *logic* is the study of these tools in order to determine differences between *valid* and *invalid* arguments, those which are worthy of use in our reasoning and those which are not.

In order to begin to study arguments, logic first classifies them according to their different forms. In order to effect this classification, it has proven useful to use a variety of symbols. The way in which logicians do this is to create a formalized language and provide for translating and abstracting from ordinary language into this formalized language. This formalized system is called *Symbolic Logic*. Just as it is easier to study the laws of arithmetic by using the abstraction of algebra, it will be easier to study reasoning by using symbolic logic to discover, for example, criteria of consistency and validity.

In summary, symbolic logic is a formal theory of arguments that studies how they are used in reasoning to arrive at conclusions. Its essence consists of abstracting the form of a statement away from its content. Within this formal abstraction, we will develop and represent logical principles through a system of primitive symbols, axioms, and rules of inference. This framework will help us identify valid argument forms that will permit us to infer valid conclusions from given premises through the concept of proof. The symbolic logical systems that we will study are those of the **Statement Calculus** and some **Predicate Calculus**.

The Statement Calculus

We will begin our study of symbolic logic by examining how reason is used in formal arguments within the Statement Calculus. In this system of logic, the truth-value of a statement is determined by the truth-values of its component statements. But first, what is a statement?

Definition: A **statement** is a declarative sentence that is either true or false but not both.

In analyzing statements, one of the first things we note is that some statements are simple while others are compound. This is seen in the following examples.

Definition: A **simple statement** is a statement that has one subject, one predicate, and is not the combination of other statements through the use of connectives like *"not"*, *"and"*, *"or"*, *"if-then"*, and *"if and only if"*. Such statements are indivisible.

Some of the following examples are based on the comic strip character Spiderman, alias, Peter Parker, with his girlfriend Jane Watson or Thor, the hammer-wielding Norse god. Notice that the simple statements consist of a single subject, which is underlined, and a single predicate, which is underlined twice.

Example Set 1:

(1) *Spiderman is a superhero.*
(2) $\sqrt{2}$ *is irrational.*
(3) *Some superheroes can fly.*
(4) *The real number π is transcendental.*
(5) *All superheroes battle crime.*
(6) *2 + 1 is 8.*
(7) *All students in this class will receive the grade "A".*
(8) *Thor battles the Frost Giants.*

We use simple statements to build compound statements.

Definition: A **compound statement** is a statement which consists of a number of simple statements joined together with one or more connectives.

Example Set 2: Notice that the following compound statements are formed from simple statements by means of the underlined connectives.

(1) *Spiderman is a superhero and he cannot fly.*
(2) *The square of an integer is odd if and only if the integer is odd.*
(3) *Spiderman is not a superhero.*
(4) *If Socrates is a man, then Socrates is mortal.*
(5) *If $\sqrt{2}$ is irrational, then $\sqrt{2}$ cannot be written as a repeating decimal.*
(6) *It is not the case that $\sqrt{2}$ is rational.*
(7) *The real number π is not the zero of any polynomial with integer coefficients.*
(8) *Thor is a superhero or he can fly.*
(5) *If Socrates is not mortal, then Socrates is a god.*
(9) *If I study hard, then I will receive an excellent grade in this course.*

Why are the following sentences not statements?

(1) *This sentence is false*

(2) $x + 7 = 3x$

Syntax of the Statement Calculus

Just as in Algebra, in which one uses variables to denote numbers and symbols, like "+" for "addition", to denote operations, and parenthesis to eliminate ambiguity, so in the Statement Calculus one uses variables to denote statements, various symbols to denote connectives, their operations, and parentheses to avoid ambiguity. That is, our lexicon will consist of variables, connectives and parentheses/brackets. The formalization of the Statement Calculus then proceeds as follows.

Definition: Symbols like "a", "b", "c", etc., and such symbols with numerical subscripts are called **statement variables** and can symbolize arbitrary simple statements.

Definition: The **connectives** that are used to form compound statements are given in the following table.

Connectives

Negation	*not ...*	$\neg\ldots$
Conjunction	*... and ...*	$\ldots\wedge\ldots$
Disjunction	*... or ...*	$\cdots\vee\cdots$
Conditional	*if ... then ...*	$\cdots\Rightarrow\cdots$
Biconditional	*... if and only if ...*	$\cdots\Leftrightarrow\cdots$

Figure 1

What is the **logical form** of a statement? First we define what we mean by an expression.

Definition: An **expression** is a finite sequence of statement variables and connectives.

For example, the sequence "$\Rightarrow pq\wedge$" is an expression but is not well formed. This sequence can never be a logical form. The expressions that can represent statements are called **logical forms, formal statements, well-formed formulas**, or, simply, **formulas**. That is,

Definition: The **logical forms**[1] of the Statement Calculus are expressions generated by the following clauses:

(a) Every statement variable is a logical form.

(b) If $\mathcal{A}, \mathcal{B}$ are logical forms, then so are $(\mathcal{A}\wedge\mathcal{B}), (\mathcal{A}\vee\mathcal{B}), (\neg\mathcal{A}), (\mathcal{A}\Rightarrow\mathcal{B})$, and $(\mathcal{A}\Leftrightarrow\mathcal{B})$.

[1] We will frequently use the term "formula" for the expression "logical form".

(c) Nothing is a logical form unless it can be obtained by repeated applications of clauses (a) and (b).

As an example, show that the expression "$(p \Rightarrow q) \Rightarrow ((\neg q) \Rightarrow (\neg p))$" is a logical form by illustrating its derivation in a decomposition tree.

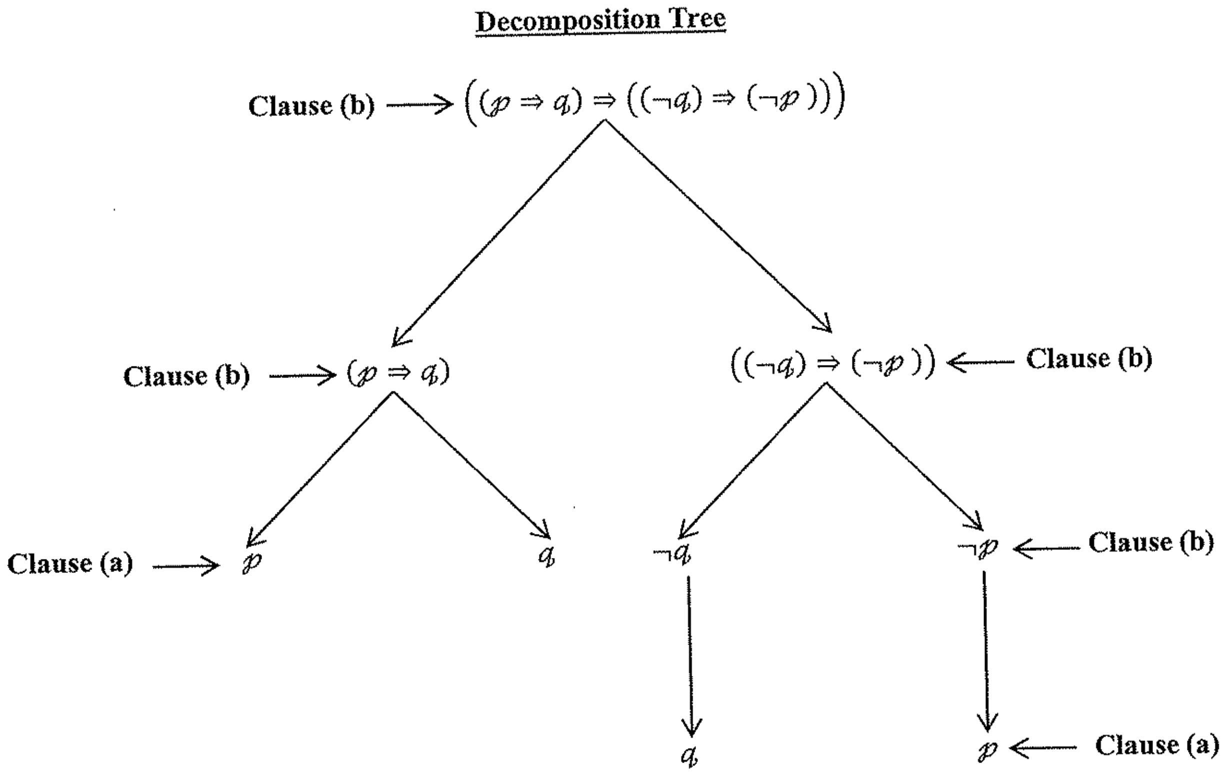

Figure 2

In this decomposition, each node is a sub-formula of the root, the given formula. Starting with the leaf nodes, the variables of the formula, each successive node is the sub-formula gotten by applying a clause to the nodes of the immediate branches below until the formula "$(p \Rightarrow q) \Rightarrow ((\neg q) \Rightarrow (\neg p))$" is reached.

We eliminate structural ambiguity in the Statement Calculus by the use of parentheses. For example, $p \Rightarrow q \vee r$ is ambiguous so that we need to write either $(p \Rightarrow q) \vee r$ or $p \Rightarrow (q \vee r)$, depending on the form we intend. Because parentheses can become cumbersome to use, we will adopt the convention that the connectives' **binding precedence** in decreasing order is as exhibited in Figure 1. That is, "$\neg$" has the highest binding precedence, "$\wedge$" the next, "$\vee$" follows, "$\Rightarrow$" the penultimate, and "$\Leftrightarrow$" the least. Logical operators of the same precedence are left associative. Therefore, the formula "$p \vee q \Leftrightarrow \neg r \Rightarrow s \Rightarrow t$" means

$$(p \vee q) \Leftrightarrow (((\neg r) \Rightarrow s) \Rightarrow t)$$

Note, if we intend the form "$(p \vee q) \wedge r$", then the parentheses must remain because "$\wedge$" has a higher binding precedence than "$\vee$".

To avoid the confusion that can arise when discussing formulas and statements, we will use the following fonts and letters to distinguish between the two:

Statements	a, b, c ...	Variables for arbitrary simple statements, from the whole alphabet
Statements	A, B, C ...	Letters naming a particular simple statement, from the whole alphabet
Formulas	$\mathcal{A}, \mathcal{B}, \mathcal{C}$ ...	Meta symbols for arbitrary formulas, from the whole alphabet

Figure3

With the above formalization, each of the above example statements has a **logical form**. That is,

Definition: The **logical form** of a statement is its symbolization. It is what remains after you remove its content.

As some examples, let us symbolize or determine the logical form of the following compound statements.

Example Set 3:

(1) If p symbolizes *Thor won the first battle* and q symbolizes *Thor won the second battle,* determine the logical form of the following compound statements.

Statements	Logical Form
(a) *Thor won both battles.*	$p \wedge q$
(b) *Thor won at least one battle.*	$p \vee q$
(c) *Thor lost both battles.*	$\neg p \wedge \neg q$
(d) *Thor lost at least one battle.*	$\neg p \vee \neg q$
(e) *Thor won at most one battle.*	$\neg p \vee (p \wedge \neg q)$
(f) *Thor lost at most one battle.*	$p \vee (\neg p \wedge q)$

Figure 3

(2) If p symbolizes *The superhero is Spiderman,* q symbolizes *The superhero will battle evil,* and r symbolizes *The superhero will wed Jane Watson,* determine the logical form of the following compound statement.

If the superhero is Spiderman, then the superhero will either battle evil and will not wed Jane Watson or the superhero will not battle evil and will wed Jane Watson.

Replacing each statement with its corresponding symbol and each verbal connective with its corresponding symbolic connective, we obtain the following logical form.

$$p \Rightarrow q \wedge \neg r \vee \neg q \wedge r$$

Recall, this means

$$p \Rightarrow \left(\left(q \wedge (\neg r)\right) \vee \left((\neg q) \wedge r\right)\right),$$

because of the binding precedence of the connectives.

Observe that if a compound statement is symbolized in this way, then only the bare logical bones are exposed, a mere *logical form,* which several different statements might have in common. It is precisely this which will enable us to analyze deduction. This is so because deduction has to do with the forms of statements in an argument rather than their meanings.

Exercise Set 1: Place the following statements in logical form.

(1) *You will receive an A if you work hard and the sun shines, or you work hard and it rains.*

(2) *The only superhero who thinks that "with great power there must also come great responsibility" is Spiderman.*

(3) *The only superhero that battles evil and his own angst of human uncertainty is Spiderman.*

(4) *The real number* π *is not the zero of any polynomial with integer coefficients.*

(5) *A thought is a great truth if and only if the thought is applicable to all men and all times.*

(6) *The natural number 4 is even or* $\sqrt{2}$ *is rational.*

(7) *He is a lawyer only if he hasn't been disbarred. It's not the case that he has not been disbarred. Therefore he is not a lawyer.*

Having explained the lexicon or grammar of the statement calculus, we will now concentrate on its semantics.

Semantics of the Statement Calculus

It is very important to observe that a logical form is never regarded as true or false. However, the logical form can become true or false under an appropriate interpretation. That is,

Definition: For a given logical form or formula, if we substitute the truth-values of statements for the statement variables in this formula, in which the same truth-value of a statement always substitutes the same variable, the resulting expression of truth-values is said to be an **instance** of the given formula. This resulting instance will have a truth-value that only depends on the substituted truth-values and the connectives that bind them. The process of assigning a truth-value to the given formula is called an **interpretation** of the given formula and will be defined by a number of tabular forms, called truth tables. In this way, we will present the semantics of the statement calculus. In general, we can assign arbitrary truth-values to the statement variables in question without going through the intermediary statement. Given a formula $\mathcal{A}$, $\mathfrak{I}(\mathcal{A})$ will denote an interpretation of $\mathcal{A}$. Observe that the truth-value of every statement is an instance of its logical form.

Before we proceed, consider the conditional formula $\mathcal{A} \Rightarrow \mathcal{B}$. Recall when the conditional statement $\mathfrak{I}(\mathcal{A}) \Rightarrow \mathfrak{I}(\mathcal{B})$ is regarded as true. The statement $\mathfrak{I}(\mathcal{A})$ is called the **hypothesis/premise** and the statement $\mathfrak{I}(\mathcal{B})$ is called the **conclusion/consequent** of the conditional. Such a statement is considered true if it is not the case that $\mathfrak{I}(\mathcal{A})$ is true and $\mathfrak{I}(\mathcal{B})$ is false; otherwise, the conditional is false.

Definition: A **truth table** of a formula is a table of rows and columns headed by the component sub-formulas of the given formula and followed last by the formula itself. In this table, each row exhibits the truth-values of one possible interpretation of each sub-formula or formula while each column under each sub-formula or formula corresponds to the truth-values of that sub-formula or formula for all possible interpretations. All the rows taken together exhibit all the possible interpretations that can occur, the logical possibilities. Thus, each row is a possible world in which our formula has been abstractly interpreted.

Let us begin by exhibiting the truth tables of all the basic connectives. That is, these truth tables constitute the semantics of our basic connectives, hence that of the statement calculus.

Remark 1: In the following tables, "T", read as "top", is to be thought of as "true", while "⊥", read as "bot", is to be thought of as "false".

If $\mathcal{A}$ and $\mathcal{B}$ are arbitrary formulas, then

Truth Table Semantics of the Basic Connectives

$\mathcal{A}$	$\neg\mathcal{A}$
T	⊥
⊥	T

Negation
Figure 4

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A} \wedge \mathcal{B}$
T	T	T
T	⊥	⊥
⊥	T	⊥
⊥	⊥	⊥

Conjunction
Figure 5

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A} \vee \mathcal{B}$
T	T	T
T	⊥	T
⊥	T	T
⊥	⊥	⊥

Disjunction
Figure 6

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A} \Rightarrow \mathcal{B}$
T	T	T
T	⊥	⊥
⊥	T	T
⊥	⊥	T

Conditional
Figure 7

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A} \Leftrightarrow \mathcal{B}$
T	T	T
T	⊥	⊥
⊥	T	⊥
⊥	⊥	T

Biconditional
Figure 8

The semantics of the above connectives as given in the above tables were determined, as best we could, from the corresponding way they are used in in our natural language, English here. Some correspond exactly while others correspond poorly. Let p symbolize *The Queen of Hearts made some tarts* and q symbolize *Little Boy Blue blew his horn.* Then,

$\neg$(Not):
The table in Figure 4 corresponds exactly to the way "not" is used in English. Thus, $\neg p$ symbolizes the statement *The Queen of Hearts did not make some tarts* and is false if and only if the statement *The Queen of Hearts made some tarts* is true.

$\wedge$ (And):
The table in Figure 5 corresponds exactly to the way "and" or "but" is used in English. Thus, $p \wedge q$ symbolizes the statement *The Queen of Hearts made some tarts and Little Boy Blue blew his horn.* This statement is true if both component statements are true and false if at least one component statement is false.

$\vee$ (Or):
However, the semantics exhibited in the table of Figure 6 corresponds to the inclusive use of the conjunction "or" in English. Thus, $p \vee q$ symbolizes the statement *The Queen of Hearts made some tarts or Little Boy Blue blew his horn.* This statement is true if at least one of the component statements is true or both of the component statements are true.

$\Rightarrow$ (If ... then ...):
The semantics of the conditional exhibited in Figure 7 requires some explanation. These semantics correspond poorly to the way this English conjunction is used. However, the first two rows of this table are exactly what we would expect. What is the explanation of the last two rows? In general, whatever the interpretations of p and q we would expect $(p \wedge q) \Rightarrow p$ to be true since the consequent is part of the antecedent. Thus, if p were true and q were false, then $(p \wedge q) \Rightarrow p$ is "false⇒true", which would then be true. This is the third row. On the other hand, if p were false and q whatever, then $(p \wedge q) \Rightarrow p$ is "false⇒false", which would then be true. This is the last row.

$\Leftrightarrow$ (If and only if):
The semantics of Figure 8 is easy to understand and corresponds to the English "if and only if". Thus, $p \Leftrightarrow q$ symbolizes the statement *The Queen of Hearts made some tarts if and only if Little Boy Blue blew his horn.* This statement is true if both component statements have the same truth-value and false if they differ.

The connectives "⇒", "∨", and "⇔" can be defined in terms of the connectives "¬" and "∧". We will do this only for the connectives "⇒" and "⇔". The reason we can do this will be seen later. Therefore,

Definition: $\mathcal{A} \Rightarrow \mathcal{B}$ is defined to be $\neg(\mathcal{A} \wedge \neg(\mathcal{B}))$.

Definition: $\mathcal{A} \Leftrightarrow \mathcal{B}$ is defined to be $(\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\mathcal{B} \Rightarrow \mathcal{A})$.

Using the above semantics of our basic connectives, we may now construct a truth table of any formula.

Example 4: Construct a truth table of the formula $((p \Rightarrow q) \wedge (\neg q)) \Rightarrow (\neg p)$.

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$(p \Rightarrow q) \wedge \neg q$	$((p \Rightarrow q) \wedge \neg q) \Rightarrow (\neg p)$
T	T	⊥	⊥	T	⊥	T
T	⊥	⊥	T	⊥	⊥	T
⊥	T	T	⊥	T	⊥	T
⊥	⊥	T	T	T	T	T

Figure 9

The tuple of truth values for the simple statements p, q in any row is referred to as a logical possibility. Notice how the truth values of the final column, the original logical form, depends on the truth values of each of all the logical possibilities. Thus, a truth table is a record of this dependency. From another point of view each logical form can be considered to be a function from the domain of all its logical possibilities to the range $\{T, \bot\}$.

Example 5: Construct a truth table of the formula $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T
T	⊥	⊥	T	⊥
⊥	T	T	⊥	⊥
⊥	⊥	T	T	T

Figure10

Example 6: Construct a truth table of the formula $\neg(p \wedge (\neg q))$.

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge (\neg q))$
T	T	⊥	⊥	T
T	⊥	T	T	⊥
⊥	T	⊥	⊥	T
⊥	⊥	T	⊥	T

Figure 11

Example 7: Construct a truth table of the formula $p \vee \neg p$.

p	$\neg p$	$p \vee \neg p$
T	⊥	T
⊥	T	T
T	⊥	T
⊥	T	T

Figure 12

Example 8: Construct a truth table of the formula $p \wedge (\neg p)$.

p	$\neg p$	$p \wedge (\neg p)$
T	⊥	⊥
⊥	T	⊥
T	⊥	⊥
⊥	T	⊥

Figure 13

Example 9: There is an alternate and more efficient way of presenting the truth table of a formula. This method is also less prone to error than the method already presented. Let us illustrate this alternative by constructing the truth table of the formula given in example 4 again.

(1)	(3)	(2)	(6)	(5)	(4)	(9)	(8)	(7)
$((p$	$\Rightarrow$	$q)$	$\wedge$	$(\neg$	$q))$	$\Rightarrow$	$(\neg$	$p)$
T	T	T	⊥	⊥	T	T	⊥	T
T	⊥	⊥	⊥	T	⊥	T	⊥	T
⊥	T	T	⊥	⊥	T	T	T	⊥
⊥	T	⊥	T	T	⊥	T	T	⊥

Figure 14

Most of mathematics is phrased in terms of $\mathcal{A} \Rightarrow \mathcal{B}$. There are several idioms that have the same meaning as "If $\mathcal{A}$, then $\mathcal{B}$".

- If $\mathcal{A}$, then $\mathcal{B}$
- $\mathcal{A}$ is a sufficient condition for $\mathcal{B}$
- $\mathcal{B}$ if $\mathcal{A}$
- $\mathcal{A}$ only if $\mathcal{B}$
- $\mathcal{B}$ is a necessary condition for $\mathcal{A}$
- $\mathcal{B}$ provided that $\mathcal{A}$
- If not $\mathcal{B}$, then not $\mathcal{A}$
- $\mathcal{A}$ implies $\mathcal{B}$

There are three types of logical forms (formulas).

Definition: A formula is a **tautology** if and only if the truth value of the formula is T for all its logical possibilities; that is, the last column corresponding to its truth table consists of all T's.

For example, the formula $p \vee \neg p$ in example 7 is a tautology. For convenience, "T" will replace the sub-formula $p \vee \neg p$ in any formula containing this tautology.

Definition: A formula is a **contradiction** if and only if the truth value is ⊥ for all its logical possibilities; that is, the last column corresponding to its truth table consists of all ⊥'s.

For example, the formula $p \wedge \neg p$ in example 8 is a contradiction. For convenience, "⊥" will replace the sub-formula $p \wedge \neg p$ in any formula containing this contradiction.

Definition: A formula is a **contingency** if and only if it is neither a tautology nor a contradiction.

For example, the formula in example 6 is a contingency.

Exercise Set 2: Determine which of the following statements are tautological and which are contingent.

(1) The President of the U.S.A. is a man.
(2) If I love you then I love you.
(3) Either all students like logic or some students don't like logic.
(4) Some roses are red.
(5) It will either rain or not rain in Hattiesburg on June 28, 2020.
(6) Some students are bored by logic.

In logic, we sometimes would like to replace one formula by an equivalent one. This idea is made precise in following definition.

Definition: Two formulas $\mathcal{A}$ and $\mathcal{B}$ are **logically equivalent** if $\mathcal{A} \Leftrightarrow \mathcal{B}$ is a tautology; that is, the last columns of the truth table of each formula are identical. In this case, we will write $\mathcal{A} \equiv \mathcal{B}$.

For example, the logical forms above in the truth tables of Figure 7 and Figure 11 are logically equivalent. This is the reason we can define the connective "⇒" in terms of the connectives "¬" and "∧".

Exercise Set 3: Construct the truth table of each of the following logical forms.

(1) $((\neg p) \wedge (\neg q))$

(2) $((p \Rightarrow q) \Rightarrow (\neg(q \Rightarrow p)))$

(3) $(p \Rightarrow (q \Rightarrow r))$

(4) $((p \wedge q) \Rightarrow r)$

(5) $((p \Leftrightarrow (\neg q)) \vee r)$

(6) $((p \wedge q) \vee (r \wedge s))$

(7) $((\neg p) \wedge q)$

(8) $((p \Rightarrow (q \Rightarrow r)) \Rightarrow (p \Rightarrow q) \Rightarrow (p \Rightarrow r))$

Exercise Set 4: Which of the following logical forms are tautologies?

(1) $(p \Rightarrow q) \Rightarrow p$

(2) $((q \vee r) \Rightarrow ((\neg r) \Rightarrow q))$

(3) $((p \wedge (\neg q)) \vee (q \wedge (\neg r)) \vee (r \wedge (\neg p)))$

(4) $((p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \wedge (\neg q)) \vee r))$

Exercise Set 5: Show that the following formulas are tautologies.

(1) $(p \Rightarrow p)$

(2) $(p \Rightarrow (q \Rightarrow p))$

(3) $p \Rightarrow (p \wedge p)$

(4) $(p \wedge q \Rightarrow q)$

(5) $(p \vee \bot) \Leftrightarrow p$

(6) $((p \wedge \top) \Leftrightarrow p)$

(7) $((p \Rightarrow q) \Leftrightarrow ((p \wedge (\neg q)) \Rightarrow \bot))$

(8) $((p \vee \top) \Leftrightarrow \top)$

(9) $(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$

(10) $(p \wedge \bot) \Leftrightarrow \bot$

(11) $(\neg p \Rightarrow \bot) \Leftrightarrow p$

(12) $\bot \Rightarrow p$

(13) $\neg p \Rightarrow \top$

(14) $(p \vee (\neg p)) \Leftrightarrow \top$

Exercise Set 6: Show the following are tautologies. These tautologies will play a central role in proving arguments and theorems by deductive reasoning. Eventually, they will be used as rules of inference.

(a) Law of Addition (Add):
$\mathcal{A} \Rightarrow (\mathcal{A} \vee \mathcal{B})$

(b) Laws of Simplification (Simp):
$(\mathcal{A} \wedge \mathcal{B}) \Rightarrow \mathcal{A}$
$(\mathcal{A} \wedge \mathcal{B}) \Rightarrow \mathcal{B}$

(c) Disjunctive Syllogism (DS):
$((\mathcal{A} \vee \mathcal{B}) \wedge (\neg \mathcal{A})) \Rightarrow \mathcal{B}$

(d) Law of Double Negation (DN):
$(\neg(\neg \mathcal{A})) \Leftrightarrow \mathcal{A}$

(e) Commutative Laws (Com):
$(\mathcal{A} \wedge \mathcal{B}) \Leftrightarrow (\mathcal{B} \wedge \mathcal{A})$
$(\mathcal{A} \vee \mathcal{B}) \Leftrightarrow (\mathcal{B} \vee \mathcal{A})$

(f) Laws of Idempotency (Idemp):
$(\mathcal{A} \wedge \mathcal{A}) \Leftrightarrow \mathcal{A}$
$(\mathcal{A} \vee \mathcal{A}) \Leftrightarrow \mathcal{A}$

(g) Contrapositive Law (Contrap):
$(\mathcal{A} \Rightarrow \mathcal{B}) \Leftrightarrow ((\neg \mathcal{B}) \Rightarrow (\neg \mathcal{A}))$

(h) DeMorgan's Laws (De M.):
$\neg(\mathcal{A} \wedge \mathcal{B}) \Leftrightarrow ((\neg \mathcal{A}) \vee (\neg \mathcal{B}))$
$\neg(\mathcal{A} \vee \mathcal{B}) \Leftrightarrow ((\neg \mathcal{A}) \wedge (\neg \mathcal{B}))$

(j) Distributive Laws (Dist):
$(\mathcal{A} \wedge (\mathcal{B} \vee \mathcal{C})) \Leftrightarrow ((\mathcal{A} \wedge \mathcal{B}) \vee (\mathcal{A} \wedge \mathcal{C}))$
$(\mathcal{A} \vee (\mathcal{B} \wedge \mathcal{C})) \Leftrightarrow ((\mathcal{A} \vee \mathcal{B}) \wedge (\mathcal{A} \vee \mathcal{C}))$

(k) Transitive Laws (Trans):
$((\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\mathcal{B} \Rightarrow \mathcal{C})) \Rightarrow (\mathcal{A} \Rightarrow \mathcal{C})$
$((\mathcal{A} \Leftrightarrow \mathcal{B}) \wedge (\mathcal{B} \Leftrightarrow \mathcal{C})) \Rightarrow (\mathcal{A} \Leftrightarrow \mathcal{C})$

(l) Constructive Dilemmas (CD):
$((\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\mathcal{C} \Rightarrow \mathcal{D})) \Rightarrow ((\mathcal{A} \vee \mathcal{C}) \Rightarrow (\mathcal{B} \vee \mathcal{D}))$
$((\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\mathcal{C} \Rightarrow \mathcal{D})) \Rightarrow ((\mathcal{A} \wedge \mathcal{C}) \Rightarrow (\mathcal{B} \wedge \mathcal{D}))$

(m) Destructive Dilemmas (DD):
$((\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\mathcal{C} \Rightarrow \mathcal{D})) \Rightarrow (((\neg \mathcal{B}) \vee (\neg \mathcal{D})) \Rightarrow ((\neg \mathcal{A}) \vee (\neg \mathcal{C})))$
$((\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\mathcal{C} \Rightarrow \mathcal{D})) \Rightarrow (((\neg \mathcal{B}) \wedge (\neg \mathcal{D})) \Rightarrow ((\neg \mathcal{A}) \wedge (\neg \mathcal{C})))$

(n) Modus Ponens (MP):
$((\mathcal{A} \Rightarrow \mathcal{B}) \wedge \mathcal{A}) \Rightarrow \mathcal{B}$

(o) Modus Tolens (MT):
$((\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\neg \mathcal{B})) \Rightarrow \neg \mathcal{A}$

(p) Contradiction (C):
$(\neg \mathcal{A} \wedge \mathcal{A}) \Leftrightarrow \bot$

(q) Reductio ad Absurdum (R.A.):

(i) Associative Laws (Assoc):

$$((\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C}) \Leftrightarrow (\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C}))$$
$$((\mathcal{A} \vee \mathcal{B}) \vee \mathcal{C}) \Leftrightarrow (\mathcal{A} \vee (\mathcal{B} \vee \mathcal{C}))$$

$$((\mathcal{A} \wedge (\neg\mathcal{B})) \Rightarrow \bot) \Leftrightarrow (\mathcal{A} \Rightarrow \mathcal{B})$$
$$((\neg\mathcal{A}) \Rightarrow \bot) \Leftrightarrow \mathcal{A}$$

Now that we know what a statement is, its syntax, semantics, and logical form, we can present the important concept of an argument.

Arguments and Their Validity

Definition: An **argument** is a sequence of statements called the **premises** and a final statement called the **conclusion.**

An argument form is what you get after you replace each statement in an argument with its logical form. More precisely, we have

Definition: An **argument form** is a sequence of logical forms called the **premises** and a final logical form called the **conclusion.**

We will represent an argument form as follows:

$$\begin{array}{c} \mathcal{P}_1 \\ \mathcal{P}_2 \\ \vdots \\ \mathcal{P}_n \\ \hline \mathcal{C} \end{array} \qquad \text{or} \qquad \mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n | \mathcal{C}$$

Figure 15

Remark 2: Usually the word "therefore", or some synonym, or shorthand symbol as ∴ (read "therefore"), is written or understood just before the conclusion. Therefore, the above line segment separating the premises from the conclusion in Figure 15 will be understood to mean *therefore.*

Definition: An argument form is **valid** if it is impossible to have an interpretation of the given formulas in such a way as to make each of the premises true and the conclusion false. Otherwise the argument form is **invalid.**

Remark 3: A mechanical way of determining if our argument form is valid is to determine if the formula $(\mathcal{P}_1 \wedge \mathcal{P}_2 \wedge \dots \wedge \mathcal{P}_n) \Rightarrow \mathcal{C}$ is a tautology. That is, construct a truth table of this formula and observe if there is a logical possibility in which the premises are true and the conclusion is false.

In other words, an argument form is valid if the conclusion must be true in any world we can imagine in which the premises are true. In this case, we also say,

Definition: The final logical form of a valid argument form is said to be a **logical consequence** of its preceding premises.

Correspondingly,

Definition: An argument is **valid** if its corresponding argument form is valid. Otherwise the argument is **invalid.**

Definition: The final statement of a valid argument is said to be a **logical consequence** of its preceding premises.

Let us illustrate these ideas of validity in the following examples. In the following arguments,

Spiderman refers to the comic book superhero by that name.
Garfield refers to the comic book cat by that name.
Socrates refers to the ancient Greek philosopher by that name.

Example 10: Consider the following argument,

If Spiderman is a superhero, then Spiderman battles evil.
Spiderman is a superhero.
___
$\therefore$ *Spiderman battles evil.*

What is the argument form of this argument? To see the answer, let p symbolize *Spiderman is a superhero* and q symbolize *Spiderman battles evil,* everywhere in our argument. Then the argument form of this argument is exhibited in Figure 16a.

$$\begin{array}{l} p \Rightarrow q \\ \underline{p \qquad} \\ \therefore q \end{array}$$

Figure 16a

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge p$	$((p \Rightarrow q) \wedge p) \Rightarrow q$
T	T	T	T	T
T	⊥	⊥	⊥	T
⊥	T	T	⊥	T
⊥	⊥	T	⊥	T

Figure 16b

Notice, for this form, it is impossible for the premises to be true and the conclusion false under any interpretation. For, if A, B are interpretations of p, q, respectively, with B false, then A must also be false in order to have $A \Rightarrow B$ true. So, the second premise cannot be true. Thus, whenever the premises $A \Rightarrow B, A$ are true, the conclusion B must also be true. In particular, whenever the premises about Spiderman are true, the conclusion about Spiderman must also be true. In fact, in this example, both the premises and conclusion of the argument are true. Thus, our argument is valid. This can also be seen from the truth table in Figure 16b of the formula $((p \Rightarrow q) \wedge p) \Rightarrow q$, since this formula is a tautology (refer to remark 3 above).

Example 11: Consider next the argument,

If all fruit are seedless, then some fruit is seedless.
All fruit are seedless.
___
$\therefore$ *Some fruit is seedless.*

What is the logical form of this argument? To see this, we let p symbolize *All fruit are seedless* and q symbolize *Some fruit is seedless,* everywhere in our argument. Then the argument form of this argument is the same as the previous one exhibited in Figure 16a above.

As before, no interpretation of this form can render the premises true and the conclusion false. In particular, whenever the premises about fruit are true, the conclusion about fruit must also be true. In fact, in this example, the second premise is false and conclusion is true. However, this argument is still valid.

Example 12: Consider next the argument,

If Socrates is a myth, then Socrates is a Greek god.
Socrates is a myth.
___
$\therefore$ *Socrates is a Greek god.*

What is the form of this argument? We begin, as before, by first letting p symbolize *Socrates is a myth* and q symbolize *Socrates is a Greek god,* everywhere in our argument. Then the argument form of this argument is the same as the previous one exhibited in Figure 16a above.

As before, no interpretation of this form can render the premises true and the conclusion false. For, if A, B are interpretations of p, q, respectively, with $A \Rightarrow B$ and A true, then B must also be true in order to maintain $A \Rightarrow B$ true. So, the conclusion must be true. Therefore, whenever the premises $A \Rightarrow B, A$ are true, the conclusion B must be true. Here, whenever the premises about Socrates are true, the conclusion about Socrates must also be true. In fact, in this example, the second premise is false and conclusion is also false. But the argument is still valid.

Example 13: Now, consider the argument,

If President Lincoln was assassinated, then President Lincoln was murdered.
President Lincoln was murdered.
―――――――――――――――――――――――――
∴ *President Lincoln was assassinated.*

What is the form of this argument? We begin, as before, by letting p symbolize *President Lincoln was assassinated* and q *symbolize President Lincoln was murdered*, everywhere in our argument. Then the argument form of this argument is given in 17a.

$p \Rightarrow q$
q
―――――
$\therefore p$

Figure 17a

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge q$	$(((p \Rightarrow q) \wedge q) \Rightarrow p)$
T	T	T	T	T
T	⊥	⊥	⊥	T
⊥	T	T	T	⊥
⊥	⊥	T	⊥	T

Figure 17b

In this example about President Lincoln, even though both the premises and the conclusion are true, notice, for this form, it is possible for the premises to be true and the conclusion false under some interpretation. For, if A, B are interpretations of p, q, respectively, with A false and B true, then $A \Rightarrow B$ must also be true. So, the premises $A \Rightarrow B, B$ would be true and the conclusion A would be false. An example of an invalid interpretation of the argument form in Figure 17a is given in the next example about Garfield.

Example 14:

If Garfield is a dog, then Garfield is a cat.
Garfield is cat.
―――――――――――――――――
∴ *Garfield is a dog.*

Thus, the arguments of example 13 and example 14 are invalid. This can also be seen from the truth table in Figure 17b of the formula $((p \Rightarrow q) \wedge q) \Rightarrow p$, since this formula is not a tautology (refer to remark 3 above).
In summary, arguments in examples 10-12 all have the same form shown in Figure 18 below while arguments in examples 12-13 have the form shown in Figure 19 below. Notice that, for the argument form shown in Figure 18, it is not possible for the premises to be true and the conclusion false while, for the argument form shown in Figure 19, it is possible for the premises to be true and the conclusion false. In the former case, the argument form is valid while in the latter case, the argument form is invalid[2].

Argument Forms

$p \Rightarrow q$
p
―――――
$\therefore q$
Valid
Figure 18

$p \Rightarrow q$
q
―――――
$\therefore p$
Invalid
Figure 19

[2] The valid form shown in Figure 18 is called *Modus Ponens* (MP) while the invalid form shown in Figure 19 illustrates the Converse Error.

In the above examples, arguments in examples 10-12 are instances of the argument form shown in Figure 18 while arguments in example 13-14 are instances of the argument form shown in Figure 19. In the argument of example 10, the statement *Spiderman battles evil* is a logical consequence of its premises. Also, in this argument, we say that the argument is **sound** while in arguments of examples 13-14 we say that these arguments are **unsound**. That is,

Definition: A sound argument is a valid argument with true premises. Otherwise, we say that the argument is **unsound.**

Important: Arguments are never described as true or false but as valid, invalid, sound or unsound while statements are described as true or false.

For the most part, logicians are not interested in the soundness of an argument. That is, they are not interested in determining the truth or falsity of the premises of arguments. This is usually an empirical matter to be decided by observing how things operate in the world. Logicians are more concerned with the internal structure of arguments as represented by their forms and the question of whether the conclusion is a logical consequence of its premises. However, to determine this logical consequence, we must consider the possible interpretations of the premises and conclusion of the argument.

There is another important concept that can shed more light on the concept of validity. Let $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n | \mathcal{C}$ be an invalid argument form. Then, there exists an interpretation $\mathcal{I}(\mathcal{P}_1), \mathcal{I}(\mathcal{P}_2), \dots, \mathcal{I}(\mathcal{P}_n), \mathcal{I}(\mathcal{C})$ such that the premises $\mathcal{I}(\mathcal{P}_1), \mathcal{I}(\mathcal{P}_2), \dots, \mathcal{I}(\mathcal{P}_n)$ are true and the conclusion $\mathcal{I}(\mathcal{C})$ is false. Therefore, each of the statements $\mathcal{I}(\mathcal{P}_1), \mathcal{I}(\mathcal{P}_2), \dots, \mathcal{I}(\mathcal{P}_n), \neg\mathcal{I}(\mathcal{C})$ is true. In this case, we say the latter collection of statements is **consistent**. On the other hand, if the argument form is valid and $\mathcal{I}(\mathcal{P}_1), \mathcal{I}(\mathcal{P}_2), \dots, \mathcal{I}(\mathcal{P}_n), \mathcal{I}(\mathcal{C})$is any interpretation, the conclusion $\mathcal{I}(\mathcal{C})$ must be a logical consequence of its premises. This means that, whenever the premises $\mathcal{I}(\mathcal{P}_1), \mathcal{I}(\mathcal{P}_2), \dots, \mathcal{I}(\mathcal{P}_n)$ are true, the conclusion $\mathcal{I}(\mathcal{C})$ must also be true. Therefore, all of the statements $\mathcal{I}(\mathcal{P}_1), \mathcal{I}(\mathcal{P}_2), \dots, \mathcal{I}(\mathcal{P}_n), \neg\mathcal{I}(\mathcal{C})$ can never be true. In this case, we say the latter collection of statements is **inconsistent**. In summary,

Definition: A collection of statements is said to be **consistent** if they are all true and **inconsistent** otherwise. As for formulas, we have

Definition: A collection of formulas is said to be **consistent** if there is an interpretation that is consistent and **inconsistent** if no such interpretation exists.

As a result, we observe the following important relationship.
Important: Thus, a collection of formulas $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_{n-1}, \mathcal{A}_n$ is consistent if the corresponding argument form $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_{n-1} | \neg\mathcal{A}_n$ is invalid. On the other hand, the collection is inconsistent if the corresponding argument form $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_{n-1} | \neg\mathcal{A}_n$ is valid.

As an example, consider the collection of formulas $p \Rightarrow q, \neg q, p$. Is this collection consistent or inconsistent? Suppose it were consistent. Therefore, there would then be an interpretation of this collection in which the corresponding statements would all be true. This would imply that q would become false and p would become true. But thus would further imply that $p \Rightarrow q$ would become false. This cannot occur. Therefore, our collection must be inconsistent. This implies that the argument form shown in Figure 20 is valid. As another example, consider the collection of formulas $p \Rightarrow q, \neg p, q$. Is this collection consistent? Yes, it is. To see this, choose any instance of p that is false and any instance of q that is true. Then, both $\neg p$ and $p \Rightarrow q$ obviously interpret to true. Thus, our collection is consistent. This implies that the argument form shown in Figure 21 is invalid[3].

Argument Forms

$p \Rightarrow q, \neg q \therefore \neg p$	$p \Rightarrow q, \neg p \therefore \neg q$
Valid	**Invalid**
Figure 20	Figure 21

[3] The valid form shown in Figure 20 is called *Modus Tolens* (MT) while the invalid form shown in Figure 21 illustrates the Inverse Error.

We have seen that in order to determine if an argument form is valid, we must analyze the different interpretations of the argument form. In order to do this, we can exhibit our analysis in a truth table. Let us now give a significant example that illustrates this idea. Determine whether the following argument is valid or not.

Example 15: *If there are no government subsidies of agriculture, then there are government controls of agriculture. If there are government controls of agriculture, there is no agricultural depression. There is either an agricultural depression or overproduction. As a matter of fact, there is no overproduction. Therefore, there are government subsidies of agriculture.*

In order to determine the argument form of this argument, we must first determine the simple statements of our argument and symbolize each with an appropriate variable. We will name a variable with the first letter of each underlined word as a mnemonic for the simple statement that contains that word. For example, the variable s will symbolize the statement *There are government subsidies of agriculture*. After these symbolizations, we obtain the argument form and its corresponding formula in Figure 22.

Argument form	Formula
$\neg s \Rightarrow c$ $c \Rightarrow \neg d$ $d \vee o$ $\neg o$ $\therefore s$	$((\neg s \Rightarrow c) \wedge (c \Rightarrow \neg d) \wedge (d \vee o) \wedge (\neg o)) \Rightarrow s$

Figure 22

The argument in example 15 is valid if the formula in Figure 22 is a tautology. The truth table of this formula is given in Figure 23.

Truth Table of $((\neg s \Rightarrow c) \wedge (c \Rightarrow \neg d) \wedge (d \vee o) \wedge (\neg o)) \Rightarrow s$

(2)	(1)	(4)	(3)	(9)	(5)	(8)	(7)	(6)	(13)	(10)	(12)	(11)	(16)	(15)	(14)	(18)	(17)
$(((\neg$	s	$\Rightarrow$	$c)$	$\wedge$	$(c$	$\Rightarrow$	$\neg$	$d))$	$\wedge$	$(d$	$\vee$	$o))$	$\wedge$	$(\neg$	$o))$	$\Rightarrow$	s
⊥	T	T	T	⊥	T	⊥	⊥	T	⊥	T	T	T	⊥	⊥	T	T	T
⊥	T	T	T	⊥	T	⊥	⊥	T	⊥	T	T	⊥	⊥	T	⊥	T	T
⊥	T	T	T	T	T	T	T	⊥	T	⊥	T	T	⊥	⊥	T	T	T
⊥	T	T	T	T	T	T	T	⊥	⊥	⊥	⊥	⊥	⊥	T	⊥	T	T
⊥	T	T	⊥	T	⊥	T	⊥	T	T	T	T	T	⊥	⊥	T	T	T
⊥	T	T	⊥	T	⊥	T	⊥	T	T	T	T	⊥	T	T	⊥	T	T
⊥	T	T	⊥	T	⊥	T	T	⊥	T	⊥	T	T	⊥	⊥	T	T	T
⊥	T	T	⊥	T	⊥	T	T	⊥	⊥	⊥	⊥	⊥	⊥	T	⊥	T	T
T	⊥	T	T	⊥	T	⊥	⊥	⊥	⊥	T	T	T	⊥	⊥	T	T	⊥
T	⊥	T	T	⊥	T	⊥	⊥	⊥	⊥	T	T	⊥	⊥	T	⊥	T	⊥
T	⊥	T	T	T	T	T	T	T	T	⊥	T	T	⊥	⊥	T	T	⊥
T	⊥	T	T	T	T	T	T	⊥	⊥	⊥	⊥	⊥	⊥	T	⊥	T	⊥
T	⊥	⊥	⊥	⊥	⊥	T	⊥	⊥	⊥	T	T	T	⊥	⊥	T	T	⊥
T	⊥	⊥	⊥	⊥	⊥	T	⊥	⊥	⊥	T	T	⊥	⊥	T	⊥	T	⊥
T	⊥	⊥	⊥	⊥	⊥	T	T	⊥	⊥	⊥	T	T	⊥	⊥	T	T	⊥
T	⊥	⊥	⊥	⊥	⊥	T	T	⊥	⊥	⊥	⊥	⊥	⊥	T	⊥	T	⊥

Figure 23

Thus, the truth table shows that the formula is a tautology and so the argument is valid.

Exercise Set 7: Show that the following arguments / argument forms are valid.

(1) $(p \wedge q) \Rightarrow r \,\|\, p \Rightarrow (q \Rightarrow r)$

(2) *If twenty-five divisions are enough, then the general will win the battle. Either three wings of tactical air support will be provided, or the general will not win the battle. Also, it is not the case that twenty-five divisions are enough and that three wings of tactical air support will be provided. Therefore, twenty-five divisions are not enough.*

(3) *If Aristides wins, then either Northern Dancer or Citation will place. If Northern Dancer places, then Aristides will not win. If Secretariat places, then Citation will not. Therefore, if Aristides wins, Secretariat will not place.*

(4) *Either logic is difficult or not many students like it. If mathematics is easy, then logic is not difficult. Therefore, if many students like logic, then mathematics is not easy.*

(5) *If prices are high, then wages are high. Prices are high or there are price controls. Also, if there are price controls, then there is no inflation. However, there is inflation. Therefore, wages are high.*

(6) *This baby is illogical. If this baby can manage a crocodile, then it is not despised. If this baby is illogical, then it is despised. Therefore, this baby cannot manage a crocodile.*

(7) *If Rudy is a duck, it will decline to waltz. If Rudy is an officer, then he does not decline to waltz. If Rudy is a chicken, then it is a duck. Therefore, if Rudy is a chicken, then it is not an officer.*

(8) *If this mango is not ripe, then it is not wholesome. If this mango was grown by farmer Brown, then it is wholesome. If this mango was grown in the shade, then it is unripe. Therefore, if this mango was grown in the shade, then it was not grown by farmer Brown.*

(9) *If he studies medicine, then he prepares to earn a good living. If he studies the arts, then he prepares to live a good life. If he prepares to earn a good living or he prepares to live a good life, then his college tuition is not wasted. His college tuition is wasted. Therefore, he studies neither medicine nor the arts.*

(10) *Either Winston is elected president of the board or both Hilbert and Luke are elected vice presidents of the board. If either Winston is elected president or Hilbert is elected vice president of the board, then David will file a protest. Therefore, either Winston is elected president of the board or David files a protest.*

The previous method demonstrating that an argument is valid was tedious and prone to error. It required sixteen rows and eighteen columns. An argument involving n statement variables would require 2^n rows and, probably, many columns. Therefore, a more efficient method of demonstrating an argument valid would be much appreciated. There is such a method, a method that mathematicians employ every day in their professional work. A method based on rules of inference and deductive reasoning, namely, the concept of proof. We now turn to these powerful ideas.

Proofs and Deductive Reasoning

A mathematical theory can be considered to be a game in which we attempt to discover new "truths" about a domain of investigation from previously known "truths" of the domain. The starting point consists of initial "truths" about our domain called **axioms** of the **domain.** Each axiom or new truth is also said to be a theorem[4]. As in any game, there are certain rules that we must follow to obtain new theorems from old theorems. These rules are called the **rules of inference**. The application of our rules of inference in proving theorems is the engine of the process called **deductive reasoning.**

For example, to prove the formula $\mathcal{A} \Rightarrow \mathcal{B}$, we must exhibit a chain of premises, axioms, definitions or previously proved theorems starting at $\mathcal{A}$ and terminating at $\mathcal{B}$ using our rules of inference. This is made more precise in the following definition.

Definition: A **formal deductive proof** or **deductive proof** or, simply, a **proof** of the argument form $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n | \mathcal{C}$ is a sequence of formulas $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{m-1}, \mathcal{A}_m, \mathcal{C}$ in which each formula $\mathcal{A}_k, 1 \le k \le m$, or $\mathcal{C}$ in the sequence is either a premise $\mathcal{P}_k$, an axiom, a theorem[5], a definition or follows from some preceding formulas in the sequence by an application of a rule of inference. In this case we write, $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n\} \vdash \mathcal{C}$ and say that $\mathcal{C}$ is **deducible from** $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n\}$.

What should we use as our rules of inference? For a rule of inference, the most important criteria is that it be impossible to deduce a conclusion that is false from premises that are true. Our rules of inference will take the form of a valid argument. In fact, among others, all tautologies can be used as such rules of inference. Therefore, we will begin by using the tautologies on pages 9 and 10 from (a)-(q) as our rules of inference, among others. We have also added others; namely, items (c), and (s)-(w). Thus,

Rules of Inference

(a) Law of Addition (Add):
$\mathcal{A} | \mathcal{A} \vee \mathcal{B}$

(b) Laws of Simplification (Simp):
$\mathcal{A} \wedge \mathcal{B} | \mathcal{A}$
$\mathcal{A} \wedge \mathcal{B} | \mathcal{B}$

(c) Law of Conjunction (Conj):
$\mathcal{A}, \mathcal{B} || \mathcal{A} \wedge \mathcal{B}$[6]

(d) Disjunctive Syllogism (DS):
$(\mathcal{A} \vee \mathcal{B}) \wedge (\neg \mathcal{A}) | \mathcal{B}$

(e) Law of Double Negation (DN):
$\neg(\neg \mathcal{A}) || \mathcal{A}$

(f) Commutative Laws (Com):
$\mathcal{A} \wedge \mathcal{B} || \mathcal{B} \wedge \mathcal{A}$
$\mathcal{A} \vee \mathcal{B} || \mathcal{B} \vee \mathcal{A}$

(g) Laws of Idempotency (Idemp):
$\mathcal{A} \wedge \mathcal{A} || \mathcal{A}$
$\mathcal{A} \vee \mathcal{A} || \mathcal{A}$

(h) Contrapositive Law (Contrap):
$\mathcal{A} \Rightarrow \mathcal{B} || (\neg \mathcal{B}) \Rightarrow (\neg \mathcal{A})$

(i) DeMorgan's Laws (De M.):
$\neg(\mathcal{A} \wedge \mathcal{B}) || (\neg \mathcal{A}) \vee (\neg \mathcal{B})$
$\neg(\mathcal{A} \vee \mathcal{B}) || (\neg \mathcal{A}) \wedge (\neg \mathcal{B})$

(j) Associative Laws (Assoc):
$(\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C} || \mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})$
$(\mathcal{A} \vee \mathcal{B}) \vee \mathcal{C} || \mathcal{A} \vee (\mathcal{B} \vee \mathcal{C})$

(m) Constructive Dilemmas (CD):
$(\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\mathcal{C} \Rightarrow \mathcal{D}) | (\mathcal{A} \vee \mathcal{C}) \Rightarrow (\mathcal{B} \vee \mathcal{D})$
$(\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\mathcal{C} \Rightarrow \mathcal{D}) | (\mathcal{A} \wedge \mathcal{C}) \Rightarrow (\mathcal{B} \wedge \mathcal{D})$

(n) Destructive Dilemmas (DD):
$(\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\mathcal{C} \Rightarrow \mathcal{D}) | ((\neg \mathcal{B}) \vee (\neg \mathcal{D})) \Rightarrow ((\neg \mathcal{A}) \vee (\neg \mathcal{C}))$
$(\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\mathcal{C} \Rightarrow \mathcal{D}) | ((\neg \mathcal{B}) \wedge (\neg \mathcal{D})) \Rightarrow ((\neg \mathcal{A}) \wedge (\neg \mathcal{C}))$

(o) Modus Ponens (MP):
$(\mathcal{A} \Rightarrow \mathcal{B}) \wedge \mathcal{A} | \mathcal{B}$

(p) Modus Tolens (MT):
$(\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\neg \mathcal{B}) | \neg \mathcal{A}$

(q) Contradiction (C):
$\neg \mathcal{A} \wedge \mathcal{A} || \perp$

(r) Reductio ad Absurdum (R.A.):
$(\mathcal{A} \wedge (\neg \mathcal{B})) \Rightarrow \perp || \mathcal{A} \Rightarrow \mathcal{B}$
$(\neg \mathcal{A}) \Rightarrow \perp || \mathcal{A}$

(s) Deduction Theorem (DT): Let Γ be a collection of formulas.
$\Gamma \cup \{\mathcal{A}\} \vdash \mathcal{B} || \Gamma \vdash \mathcal{A} \Rightarrow \mathcal{B}$

(t) Universal Instantiation (UI):
$(\forall x \in U)\mathcal{A}(x) | \mathcal{A}(a)$, where a can be any member of U

(u) Existential Instantiation (EI):
$(\exists x \in U)\mathcal{A}(x) | \mathcal{A}(a)$, where a is some member of U not already in use.

(v) Existential Generalization (EG):
$\mathcal{A}(a) | (\exists x \in U)\mathcal{A}(x)$, where a is some member of U

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[4] A theorem can be thought of as a valid argument.

[5] A formula can be thought of an argument form with no premises. A theorem is a formula that is deducible from the empty set of formulas (has a proof). In general, a theorem is a previously proved argument / argument form.

[6] The symbol '||' here means we have two rules of inference: $\mathcal{A}, \mathcal{B} | \mathcal{A} \wedge \mathcal{B}$ and $\mathcal{A} \wedge \mathcal{B} | \mathcal{A}, \mathcal{B}$

(k) Distributive Laws (Dist):

$\mathcal{A} \wedge (\mathcal{B} \vee \mathcal{C}) || (\mathcal{A} \wedge \mathcal{B}) \vee (\mathcal{A} \wedge \mathcal{C})$

$\mathcal{A} \vee (\mathcal{B} \wedge \mathcal{C}) || (\mathcal{A} \vee \mathcal{B}) \wedge (\mathcal{A} \vee \mathcal{C})$

(l) Transitive Laws (Trans):

$(\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\mathcal{B} \Rightarrow \mathcal{C}) | \mathcal{A} \Rightarrow \mathcal{C}$

$(\mathcal{A} \Leftrightarrow \mathcal{B}) \wedge (\mathcal{B} \Leftrightarrow \mathcal{C}) | \mathcal{A} \Leftrightarrow \mathcal{C}$

(w) Universal Generalization (UG):

$\mathcal{A}(a) | (\forall x \in U)\mathcal{A}(x)$, where a is an arbitrary member of U

Figure 24

One of the most famous rules of inference is modus tolens, item (p) of Figure 24. How would this rule be used? Since it is impossible to have both $\mathcal{A} \Rightarrow \mathcal{B}, \neg\mathcal{B}$ true and $\neg\mathcal{A}$ false, in any proof, if it contains the formulas $\mathcal{A} \Rightarrow \mathcal{B}$ and $\neg\mathcal{B}$, the formula $\neg\mathcal{A}$ may be asserted as the next step. As an example of doing such a proof, we will redo example 14 above.

Example 16: Prove the theorem $((\neg s \Rightarrow c) \wedge (c \Rightarrow \neg d) \wedge (d \vee o) \wedge (\neg o)) \Rightarrow s$.

Deductive Steps	Reasons
1. $\neg s \Rightarrow c$	1;Hyp ←Hypothesis
2. $c \Rightarrow \neg d$	2;Hyp ←Hypothesis
3. $d \vee o$	3;Hyp ←Hypothesis
4. $\neg o$	4;Hyp ←Hypothesis
5. $o \vee d$	3;Com ←item (f)
6. d	4,5;DS ←item (d)
7. $\neg(\neg d)$	6;DN ←item (e)
8. $\neg c$	2,7;MT ←item (p)
9. $\neg(\neg s)$	1,8;MT ←item (p)
10. s	9;DN ←item (e)

Figure 25

The proof consists of two columns labeled "Deductive Steps" and "Reasons". The column headed "Deductive Steps" consists of the steps in our proof, from hypotheses to conclusion. In this example, the column headed "Reasons" consists of only applications of the rules of inference to the deductive steps. For example, in step 6, d is asserted as a deductive step of the proof and in the corresponding column of reasons, the rule of inference "Disjunctive Syllogism" is applied to steps 4 and 5 to obtain the deductive step 6. All this, is abbreviated as "4,5; DS". Some deductive steps are so obvious that, in the future, they will not be explicitly written in our proof but just used. For example, the commutative laws will be used in the future without being explicitly written as was done in step 5 in our proof.

In general, commentary such as "step ←item (x)" is not part of the proof. It is provided here this time only so that you can easily locate the rule of inference used in the list of the rules of inferences. It will not be supplied any more.

Remark 4: There are two major types of proofs. It depends on whether or not the proof uses the rule of inference "Reductio ad Absurdum (RA)", item (q) in Figure 24. This rule of inference is also called "Proof by Contradiction". If the proof does not use this rule, then the proof is said to be done **directly**; otherwise, it is said to be done **indirectly**. Examples 17 and 18 illustrate these two different methods of proof.

Example 17: Prove the theorem $((p \vee q) \wedge (\neg q \vee r)) \Rightarrow (p \vee r)$ directly.

Deductive Steps	Reasons

1. $p \vee q$	1;Hyp
2. $\neg q \vee r$	2;Hyp
3. $\neg q \vee \neg(\neg r)$	2;DN
4. $\neg(q \wedge (\neg r))$	3;DeM
5. $q \Rightarrow r$	4;Def of "$\Rightarrow$"
6. $\neg(\neg p) \vee \neg(\neg q)$	1;DN
7. $\neg((\neg p) \wedge (\neg q))$	6;DeM
8. $\neg p \Rightarrow q$	7;Def of "$\Rightarrow$"
9. $\neg p \Rightarrow r$	8,5;Trans
10. $\neg((\neg p) \wedge (\neg r))$	9;Def of "$\Rightarrow$"
11. $\neg(\neg p) \vee \neg(\neg r)$	10;DeM
12. $p \vee r$	11;DN

Figure 26

We will redo example 16 using proof by contradiction.

Example 18: Prove the theorem $((p \vee q) \wedge (\neg q \vee r)) \Rightarrow (p \vee r)$ indirectly.

This means we must prove the theorem $((p \vee q) \wedge (\neg q \vee r) \wedge (\neg(p \vee r))) \Rightarrow \perp$

Deductive Steps	Reasons
1. $p \vee q$	1; Hyp
2. $\neg q \vee r$	2; Hyp
3. $\neg(p \vee r)$	3; Hyp
4. $\neg p \wedge \neg r$	3; DeM
5. $\neg p$	4; Simp
6. $\neg r$	4; Simp
7. q	1,5.; DS
8. $\neg(\neg q)$	7; DN
9. r	2,8; DS
10. $\neg r \wedge r$	9,6; Conj
11. $\perp$	10, C

Figure 27

Exercise Set 8: Prove the theorems in exercise set 7 using the rules of inference.

The rules of inference from (s)-(w) involve what are called quantifiers. This brings us to the Predicate Calculus. All the examples we have exhibited above have been concerned with the formal structure of arguments only in so far as it can be expressed by our representation of simple statements such as "snow is white" and compound statements composed of one or more simple statements with one or more connectives. We use symbols like p and q to represent simple statements so we can determine general features of statements and arguments as consistency and validity. We can use truth tables in many cases to do this. However, consider the form of the following argument.

All cats are mammals
Molly is a cat
Molly is a mammal

Because the statements expressed in the premises and conclusion are all simple, we have to represent the argument as $p, q \mid r$. This argument is intuitively valid but the statement calculus has no process by which the conclusion can be derived from the premises. This occurs because this argument has internal structure the statement calculus cannot handle. In order to represent this internal structure of statements, we must use the concepts of the Predicate Calculus.

Quantification Theory of the Predicate Calculus

In the discussions of any endeavor, such as scientific studies, we are always asserting that certain individuals from a certain collection have a certain property. For example,

(a) Some lions are gentle.
(b) All birds cannot fly.
(c) All real numbers have a non-negative square.

In (a), the collection of interest is that of all lions and the asserted property[7] is "being gentle".
In (b), the collection of interest is that of all birds and the asserted property is "being able to fly".

In (c), the collection of interest is that of all reals $\mathbb{R}$ and the asserted property is "having a non-negative square root".

We now wish to develop a formalism that will help us reason logically about these assertions. The collection of interest is called the **domain of discourse** or **universe**, denoted by $\mathcal{U}$[8].

If $x \in \mathcal{U}$, and $\mathcal{P}$ is some property of interest about x, then asserting $\mathcal{P}(x)$, called a **statement predicate**, is to assert "the individual denoted by $x \in \mathcal{U}$ has property $\mathcal{P}$".

Two very important assertions are built from $\mathcal{P}(x)$ and these two phrases are: "there exist an x in $\mathcal{U}$" and "for all x in $\mathcal{U}$".

They are,

(a) "There exist an x in $\mathcal{U}$ such that $\mathcal{P}(x)$ holds"

(b) "For all x in $\mathcal{U}$, $\mathcal{P}(x)$ holds"

Statement (a) is symbolized by

$$(\exists x \in \mathcal{U})\mathcal{P}(x) \text{ or } (\exists x)(x \in \mathcal{U} \wedge \mathcal{P}(x))$$

This is an existential quantified statement and $(\exists x)$ or $(\exists x \in \mathcal{U})$ is called an **existential quantifier**.

Statement (b) is symbolized by

$$(\forall x \in \mathcal{U})\mathcal{P}(x) \text{ or } (\forall x)(x \in \mathcal{U} \Rightarrow \mathcal{P}(x))$$

This is a universally quantified statement and $(\forall x \in \mathcal{U})$ or $(\forall x)$ is called the **universal quantifier**.

Example 19: Formalize the statement: *Some lions are gentle*. This statement is an existentially quantified statement. Here, $x \in \mathcal{U}$ if and only if "x is a lion". Also, $\mathcal{G}(x)$ holds if and only if "x is gentle". Therefore, the formalized statement is $(\exists x)(x \in \mathcal{U} \wedge \mathcal{G}(x))$.

Example 20: Formalize the statement: *All real numbers have a non-negative square*. This statement is a universally quantified statement. Here, $\mathcal{U} = \mathbb{R}$. Also, $\mathcal{P}(x)$ holds if and only if $x^2 \geq 0$. Therefore, the formalized statement is $(\forall x)(x \in \mathbb{R} \Rightarrow x^2 \geq 0)$.

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[7] The asserted property is also called a predicate.
[8] A collection or set is an undefined concept that represents our intuitive understanding of the grouping together of designated elements or members. It is a many that is a one. So, if x is a member of $\mathcal{U}$, we write $x \in \mathcal{U}$.

Exercise Set 9: Formalize the following statements using quantifiers.

1. Some integer is larger than 23.
2. A positive integer is not negative,
3. No natural number is less than 0.
4. No positive integer is less than 1
5. No prime is number is smaller than 2.
6. The product of two positive integers is positive.
7. The product of two negative integers is positive.
8. The product of a positive and a negative integer is negative.
9. The sum of two even integers is even.
10. Every even integer is twice some integer.
11. Ever odd integer is one more than twice some integer.
12. The square of an even integer is even.
13. The square root of an even squared integer is even.
14. The square of an odd integer is odd.
15. The square root of an odd squared integer is odd.
16. The square root of a positive real number less than 1 is larger than the number.
17. The square root of a positive real number greater than 1 is less than the number.
18. The square root of a positive real number is positive.
19. The square root of a negative real number is not a real number.
20. The square root of a negative real number is the product of i and the square root of the absolute value of the number.

Exercise Set 10: Formalize the following statements using quantifiers.

1. For every nonzero real number there exists a nonzero real number such that the product is 1.
2. There exists a nonzero real number for every nonzero real number such that the product is 1.
3. Between every pair of distinct rational numbers there is some rational number.
4. Between every pair of distinct rational numbers there is some irrational number.
5. Between every pair of distinct real numbers there is a rational number and an irrational number.
6. Every positive integer greater than two can be written as the sum of two primes.
7. You can fool some of the people all the time.
8. You can fool all the people some of the time.
9. You can't fool all the people all the time.
10. You can't fool some person all the time.
11. Everybody likes somebody.
12. Somebody likes somebody.
13. Everybody likes everybody.
14. Somebody likes everybody.
15. Nobody likes everybody.
16. Somebody likes nobody.
17. There are exactly two purple mushrooms.
18. The barber shaves all those and only those who do not shave themselves.

Rules of Quantifier Negation:

$$\neg(\forall x \in \mathcal{U})\mathcal{P}(x) \Leftrightarrow (\exists x \in \mathcal{U})(\neg\mathcal{P}(x))$$

$$\neg(\exists x \in \mathcal{U})\mathcal{P}(x) \Leftrightarrow (\forall x \in \mathcal{U})(\neg\mathcal{P}(x))$$

Exercise Set 11:

(1) Use quantifiers to determine which of the following is logically equivalent to the negation of the statement "All snakes are poisonous"? What is the universal set?

(a) All snakes are not poisonous.
(b) Some snakes are poisonous.
(c) Some snakes are not poisonous.

(2) Find the statements that are logically equivalent to the negation of each of the following statements by first expressing each as a quantified formula and then taking the negation.
   (a) All snakes are reptiles.
   (b) Some horses are gentle.
   (c) All female students are either attractive or smart.
   (d) No baby is not cute.
   (e) No elephant can fly.

More Material

8. A detective established that one person in a gang comprised of 4 members A, B, C and D killed a person named E. The detective obtained the following statements from the gang members (S_A denotes the statement made by A. Likewise for S_B, S_C and S_D)

(i)
S_A: B killed E.
(ii)
S_B: C was shooting craps with A when E was knocked off.
(iii)
S_C: B didn't kill E.
(iv)
S_D: C didn't kill E.

The detective was then able to conclude that all but one were lying. Can you decide who killed E?

Solution Let (1)--(4) to be given later on be 4 statements. The 1st two, (1) and (2) below, are true due to the detective's work.

(1):
Only one of the statements S_A, S_B, S_C, S_D. is true
(2):
One of A, B, C and D killed E.

From the (content of the) statements S_A and S_C we know

(3):
$S_A \rightarrow S_D$ is true because if S_A is true, then B killed E which implies C didn't kill E due to (2), implying S_D is also true.

and from (1)

(4):
$S_A \rightarrow \sim S_B \wedge \sim S_C \wedge \sim S_D$ is true.

Let us examine the following sequence of statements.

(a) $S_A \rightarrow S_D$	S_A true implies S_D true, i.e. $(S_A, \therefore S_D)$ (from (3))
(b) $S_A \rightarrow \sim S_B \wedge \sim S_C \wedge \sim S_D$	S_A true implies S_B, S_C and S_D all false (from (4))
(c) $\sim S_B \wedge \sim S_C \wedge \sim S_D \rightarrow \sim S_D$	conjunctive simplification (direct)
(d) $S_A \rightarrow \sim S_D$	hypothetical syllogism (from (b), (c))
(e) $\sim S_D \rightarrow \sim S_A$	modus tollens (from (a))
(f) $S_A \rightarrow \sim S_A$	hypothetical syllogism (from (d), (e))

We note all the statements on the sequence apart from the first two (a) and (b) are obtained from their previous statements or form the valid argument forms. However the first 2 statements (a) and (b) are both true hence the conclusion in (f) is also true. A statement sequence of this type is sometimes called a **proof sequence** with the last entry called a **theorem**. The whole sequence is called the **proof** of the theorem.

Alternatively sequence (a)--(f) can also be regarded as a valid argument form in which a special feature is that the truth of the first 2 statements will ensure that all the premises there are true.

From (3) and (4) and (a)--(f) we conclude $S_A \to \sim S_A$ is true. Hence S_A must be false from the rule of contradiction (if S_A were true then $\sim S_A$ would be true, implying S_A is false: contradiction).

From the definition of S_C we see $\sim S_A \to S_C$. From the modus ponens

$$\sim S_A \to S_C, \sim S_A, \quad \therefore S_C$$

we conclude S_C is true. Since (1) gives

$$S_C \to \sim S_A \wedge \sim S_B \wedge \sim S_D,$$

we obtain from the (conjunctive simplification) argument

$$S_C \to \sim S_A \wedge \sim S_B \wedge \sim S_D, \sim S_A \wedge \sim S_B \wedge \sim S_D \to \sim S_D, \quad \therefore S_C \to \sim S_D$$

that $S_C \to \sim S_D$. Finally from modus ponens $(S_C \to \sim S_D, S_C, \quad \therefore \sim S_D)$, we conclude $\sim S_D$ is true, that is, C killed E. We note that in the above example, we have deliberately disintegreted our argument into smaller pieces with mathematical symbolisation. It turns out that verbal arguments in this case are much more concise. For a good comparison, we give below an alternative solution.

9. Re-do the previous qestion more directly.

Solution (alternative for example 6) Suppose A wasn't lying, then A's statement B killed E is true. Since A spoke the truth means B, C and D would be lying, hence the statement C didn't kill E said by D would be false, implying C did kill E. But this is a contradiction to the assumption A spoke the truth. Hence A was lying, which means B didn't kill E, which in turn implies C spoke the truth. Since only one person was not lying, D must have lied. Hence Cdidn't kill E is false. Hence C killed E.