Absolute Value Properties of the Real Numbers

For $a, x, y \in \mathrm{P}$,

Def: $|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$

Prop. 1: $|x| \geq 0$

Prop. 2: $x^2 = |x|^2$

Prop. 3: Let $a \geq 0$. $|x| = a \Leftrightarrow (x = a) \vee (x = -a)$.

Prop. 4: Let $a > 0$. $|x| < a \Leftrightarrow -a < x < a$.

Prop. 5: $|x| > a \Leftrightarrow (x < -a) \vee (x > a)$.

Prop. 6: Triangle Inequality: $|x + y| \leq |x| + |y|$.

Prop. 7: Reverse Triangle Inequality: $|x + y| \geq \big||x| - |y|\big|$.

Exercises:

1. If $0 < |x-3| < 1$, determine $? < |x+5| < ?$
2. For some $\delta > 0$, if $|x-a| < \delta_1, |y-b| < \delta_2$, determine how the following are related[3].
   (a) $|x-a|$ and $|x^2 - a^2|$
   (b) $|x-a|$ and $\left|\frac{1}{x} - \frac{1}{a}\right|$
   (c) $|x-a|$ and $|\sqrt{x} - \sqrt{a}|$, $a \geq 0$
   (d) $|x-a|, |y-b|$ and $|(x+y)-(a+b)|$
   (e) $|x-a|, |y-b|$ and $|(x \cdot y)-(a \cdot b)|$
   (f) $|x-a|, |y-b|$ and $\left|\frac{x}{y} - \frac{a}{b}\right|$

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[3] For example, in (a), determine $|x-a| < \delta_1 \Rightarrow |x^2 - a^2| < ?$