Departamento de Matemáticas
Recinto de Río Piedras

Topology Qualifying Exam   January 2014

Do exactly **five** *of the following problems. In order to obtain credit you must show all your work.*

Problem 1. Let U be a connected open set in $\mathbb{R}^n$. Then any two points in U can be connected by a curve. Define $\operatorname{dist}(p, q)$ to be the infimum of the lengths of all such curves joining points p and q in U. It can be shown that dist is a metric on U.

(i) (*10 points*) Show that "dist" defines the same topology as D, where D is the usual metric on $\mathbb{R}^n$.

(ii) (*10 points*) Show that $D = \operatorname{dist}$ if and only if the closure of U is convex.

Problem 2. (*20 points*) Suppose $(X, \mathcal{T})$ is a topological space where the set X is countable and the topology $\mathcal{T}$ is Hausdorff. Suppose further that each point of X is a limit point of X. Show that $(X, \mathcal{T})$ cannot be compact.

Problem 3. (*20 points*) In this problem, we define a topology for the set $\mathbb{R}$ that is different from the usual topology. For each $x \in \mathbb{R}$, and each real number $\epsilon > 0$, let $V(x, \epsilon) = \{x\} \cup \{q \in \mathbb{Q} : \|x - q\| < \epsilon\}$. Let $B =$ the set of all $V(x, \epsilon)$, i.e. $B = \{V(x, \epsilon) : x \in \mathbb{R}, \epsilon > 0\}$. The set B is a basis for a topology, $\mathcal{T}$, on $\mathbb{R}$. Show that $\mathcal{T}$ is strictly finer than the standard topology on $\mathbb{R}$.

Problem 4. (*20 points*) Let $Y = \{(x, y) \in \mathbb{R}^2 | (x, y) \notin \mathbb{Q} \times \mathbb{Q}\}$. Prove that Y is connected.

Problem 5. Define $f : (\mathbb{R}, \mathcal{T}_l) \to (\mathbb{R}, \mathcal{T}_{\mathcal{E}^1})$ by $f(x) = x$, $\forall x \in \mathbb{R}$. Recall, $\mathcal{T}_l$ is the lower limit topology over $\mathbb{R}$ generated by the basis $\mathfrak{B} = \{[a, b) \mid a, b \in \mathbb{R}, a < b\}$.

(i) (*6 points*) Show that f is continuous.

(ii) (*7 points*) Find a set $A \subseteq \mathbb{R}$ with $f(\overline{A}) \neq \overline{f(A)}$.

(iii) (*7 points*) Find a set $B \subseteq \mathbb{R}$ with $\overline{f^{-1}(B)} \neq f^{-1}(\overline{B})$.

Problem 6. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces and suppose $X_1 \times X_2$ has the product topology. For each $i = 1, 2$ let $A_i \subseteq X_i$. Prove that:

(i) (*10 points*) $\overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$.

(ii) (*10 points*) $\texttt{int}\,(A_1 \times A_2) = \texttt{int}\,(A_1) \times \texttt{int}\,(A_2)$.

Problem 7. Let $f, g : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be continuous functions, where $(X, \mathcal{T}_X)$ is an arbitrary topological space and $(Y, \mathcal{T}_Y)$ is a Hausdorff space.

(i) (*10 points*) Define $A = \{x \in X : f(x) = g(x)\}$. Show that A is closed.

(ii) (*10 points*) Suppose that $B \subseteq X$ is non empty such that $f(b) = g(b)$ for all $b \in B$. Show that $f(x) = g(x)$ for all $x \in \overline{B}$.