Departamento de Matemáticas
Recinto de Río Piedras

Topology Qualifying Exam — February 2010

Do exactly **five** *of the following problems. In order to obtain credit you must show all your work.*

1. Let $(X, \mathcal{T}_X)$ be a path-connected topological space. Show that $Y = \prod_{n=1}^{\infty} X_n$ where $X_n = X$ for all n, with the product topology, is also path-connected.

2. Let (Y, d) be a totally disconnected compact metric space and let C be the Cantor set. Show that there is an injective continuous map $f : Y \to C$. (Totally disconnected means that each connected component contains only one point.)

3. Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a function between compact Hausdorff spaces. Let

$$G_f = \{(x, y) \in X \times Y \mid f(x) = y\},$$

be the graph of f. Show that if G_f is closed in $X \times Y$, then f is continuous.

4. Let $X = [0, 1] \times [0, 1]$ be with the standard metric. Show that there are countably many closed subsets $S_1, S_2, \ldots, S_n, \ldots$ such that for any closed subset $Y \subseteq X$, and any $\epsilon > 0$, there is an S_n satisfying

$$\text{dist}(y, S_n) < \epsilon \quad \forall y \in Y \qquad \text{and} \qquad \text{dist}(x, Y) < \epsilon \quad \forall x \in S_n$$

5. (a) Show that $X = \{1, 2\}$ with the trivial topology is not metrizable.
   (b) Show that $\mathbb{R}_\ell$, the real numbers with the lower limit topology is not metrizable.

6. Regard the real vector space $\ell^\infty = \{(x_1, x_2, \ldots, x_n, \ldots) \mid x_i \in \mathbb{R} \text{ and } \exists M, |x_i| \leq M\}$ as a subspace of the infinite product $X = \prod_{n=1}^{\infty} X_n$ where $X_n = \mathbb{R}$ for all n (with the product topology, of course). Let $f : \ell^\infty \to \mathbb{R}$ be a continuous linear map (i.e. $f(ax + by) = af(x) + bf(y) \quad \forall a, b \in \mathbb{R}, \forall x, y \in \ell^\infty$). Show that there are finitely many real numbers $a_1, a_2, \ldots, a_n$ such that

$$f(x_1, x_2, \ldots, x_n, \ldots) = a_1 x_x + a_2 x_2 + \cdots + a_n x_n.$$

7. Let (X, d) be a non-empty compact metric space, and let $f : X \to X$ be a continuous map such that whenever $x \neq y$ we have $d(f(x), f(y)) < d(x, y)$. Show that f has a unique fixed point. (i.e. there is a unique $x \in X$ with $f(x) = x$.)