

Universidad de Puerto Rico
Departamento de Matemáticas

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M.S. Qualifying Examination
Real Variables:

Solve any **three** of the following five problems
(each problem has two parts).

- (1) Prove that the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \sin \sqrt{|x|}$ is uniformly continuous.
- (2) Prove that $\lim_{R \rightarrow \infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-R \cos x} dx = 0$.
- (3) Prove that the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \sin x^2$ is not uniformly continuous.
- (4) Prove that for each $x_1, x_2, \dots, x_n \geq 0$, the following holds:
$$(x_1 x_2 \dots x_n)^{\frac{1}{n}} \leq \frac{1}{n} (x_1 + x_2 + \dots + x_n).$$
- (5) Consider the series of real numbers: $\sum_{k=1}^{\infty} (\frac{\pi}{2} - \tan^{-1} k)$. Answer the following questions.
 - (a) Is the series convergent?
 - (b) Is the series uniformly convergent?

- (1) Prove that the function f given by $f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & x > 0 \\ 0, & x = 0 \end{cases}$ is Lebesgue integrable on any interval $[0, R]$, $R > 0$.
- (2) Suppose $f : [a, b] \rightarrow \mathbb{R}$ is Lebesgue integrable, and $f \geq 0$ a.e. on $[a, b]$. Prove that if $\int_a^b f = 0$, then $f = 0$ a.e. on $[a, b]$.
- (3) Let $I = [a, b]$ with $a < b$. Prove that $m^*(I) = b - a$. (Recall that m^* is the Lebesgue outer measure on \mathbb{R}).
- (4) Suppose E_1, E_2 are measurable subsets of \mathbb{R} . Prove that $E_1 \cup E_2$ and $E_1 \cap E_2$ are measurable.
- (5) Let $f : [a, b] \rightarrow (-\infty, \infty)$ is called lower semicontinuous at x_0 if given any $\varepsilon > 0$, one can find $\delta > 0$ such that $f(x_0) \leq f(x) + \varepsilon$ for all $x \in [a, b]$ with $|x - x_0| < \delta$. We say that f is lower semicontinuous on $[a, b]$ if it is lower semicontinuous at every $x_0 \in [a, b]$. Suppose f is lower semicontinuous on $[a, b]$. Prove the following:
- (a) There exists $M \in \mathbb{R}$ such that $f(x) \geq M$ on $[a, b]$.
 - (b) f attains its minimum on $[a, b]$.