Universidad de Puerto Rico
Departamento de Matemáticas

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REAL VARIABLES
MS Qualifying Exam
Solve any three of the following five problems
You have up to three hours.

(1) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a Lebesgue integrable function. Prove that $\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0.$

(2) Find all continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x)$ is irrational if and only if x is rational. Justify your answer.

(3) Let (f_n) (defined on $[a, b]$) be a sequence of (real valued) Lebesgue measurable functions such that $|f(x)| \leq M$ almost everywhere, where M is a positive constant. Suppose that (f_n) converges in measure to f. Show that $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$

Note. The sequence (f_n) converges in measure to f if for every $\varepsilon > 0$, $\lim_{n\to\infty} m(\{x : |f_n(x) - f(x)| > \varepsilon\}) = 0$, where m is the Lebesgue measure.

(4) Let $f : \mathbb{R} \longrightarrow [0, \infty)$ be a Lebesgue integrable function satisfying $\int_{\mathbb{R}} f = 0$. Prove that $f = 0$ almost everywhere.

(5) (a) Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent sequence of (strictly) positive real numbers, then the series $\sum_{n=1}^{\infty} (a_n)^{\frac{n}{n+1}}$ is convergent.

(b) Suppose $f : (a, b) \longrightarrow \mathbb{R}$ is a uniformly continuous function. Show that there exists a continuous function $g : [a, b] \longrightarrow \mathbb{R}$ such that $f(x) = g(x)$, $x \in (a, b)$.