**Universidad de Puerto Rico -Departamento de Matemáticas**

REAL VARIABLES- MS Qualifying Exam

Solve any three of the following five problems

**Note:** On $\mathbb{R}$, $m(.)$ and $m^*(.)$ denote the Lebesgue measure and outer measure, respectively.

(1) Let $E \subset \mathbb{R}$. Show that the following three statements are equivalent.
   (a) $E$ is measurable
   (b) For every $\varepsilon > 0$, there exists $\Omega$ open such that $E \subset \Omega$ and $m^*(\Omega \setminus E) < \varepsilon$
   (c) For every $\varepsilon > 0$, there exists $F$ closed such that $E \supset F$ and $m^*(E \setminus F) < \varepsilon$.

(2) Let $f$ be a bounded function on $[a, b]$. Prove that if $f$ is Riemann integrable, then the set of points of discontinuity of $f$ has measure zero.

(3) (a) Suppose $f_n : [0, 1] \longrightarrow \mathbb{R}$ are measurable and $\displaystyle\int_0^1 |f_n| \leq \frac{1}{n^2}$. Show that $(f_n)$ converges almost everywhere to 0.

   (b) Prove that the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(\ln k)^{\ln k}}$ converges.

(4) (a) Suppose $H$ is a nonempty compact subset of a metric space $X$. Let $f_n : H \longrightarrow \mathbb{R}$ be sequence of continuous functions such that for all $k$, $f_k(x) \leq f_{k+1}(x)$, $x \in H$. Assume that $(f_k)$ converges pointwise to $f$ and $f$ is continuous. Prove that $(f_k)$ converges uniformly.
   *Hint.* By pointwise convergence, if $\varepsilon > 0$, then for each $x \in H$, there exists $N_x > 0$ such that $k \geq N_x$ implies $|f_k(x) - f(x)| < \varepsilon/3$. Express continuity of $f_{N_x}$ and $f$ and obtain obtain an appropriate cover of $H$.

   (b) Suppose the sequence $(a_n)$ is positive and decreasing. Prove that if the series $\displaystyle\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges

(5) Let $(E_k)$ be a sequence of measurable subsets of $\mathbb{R}$ such that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Show that almost all $x \in \mathbb{R}$ lies in at most finitely many of the sets $E_k$.