Universidad de Puerto Rico
Departamento de Matemáticas
Facultad de Ciencias Naturales
Recinto de Río Piedras

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PhD Qualifying Exam: Analysis

3 hours

You may solve all seven (7) Problems (each worths 20 points) but only the best five (5) solutions will be counted as your grade. The passing grade is 70 points.

1. Let f be analytic on a region $U \subseteq \mathbb{C}$, and let $z_0 \in U$ with $f'(z_0) \neq 0$. Prove that

$$\frac{2\pi i}{f'(z_0)} = \int_\gamma \frac{1}{f(z) - f(z_0)}\, dz,$$

where γ is a small circle centered at z_0.

2. Let $(a_n)_{n\in\mathbb{N}}$ be a real valued sequence such that

$$a_1 \geq 0,\ a_2 \geq 0, \quad \text{and} \quad a_{n+2} = (a_n a_{n+1})^{1/2} \quad \text{for} \quad n \in \mathbb{N}.$$

(a) Show that (a_n) is convergent.

(b) Show that $\lim_{n\to\infty} a_n = (a_1 a_2^2)^{1/3}$.

3. Let $0 \leq u \in L^1(\mathbb{R})$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative functions in $L^1(\mathbb{R})$. Assume that (u_n) converges to u almost everywhere and $\lim_{n\to\infty} \int_{\mathbb{R}} u_n(x)\, dx = \int_{\mathbb{R}} u(x)\, dx$. Show that u_n converges to u in $L^1(\mathbb{R})$, that is, $\lim_{n\to\infty} \int_{\mathbb{R}} |u_n(x) - u(x)|\, dx = 0$.

4. Let $f(x) = \int_{\mathbb{R}} (1 + t^2)^{-1} e^{2\pi i x t}\, dt$.

(a) Show that f is bounded on $\mathbb{R}$.

(b) Calculate $f(x)$ by the method of residues.

5. (a) Suppose $f:\ \mathbb{R} \to \mathbb{R}$ is differentiable on all of $\mathbb{R}$ and $\lim_{x\to\infty} f'(x) = A$, where A is a real number. Show that $\lim_{x\to\infty} \frac{f(x)}{x}$ exists and equals A. **Hint: Show this for $A = 0$ first.**

(b) Let $f:\ [1,\infty) \to [0,\infty)$ be a non-increasing function. Prove that

$$\int_1^\infty f(x)\, dx < \infty \quad \text{if and only if} \quad \sum_{k=0}^{\infty} 2^k f(2^k) < \infty.$$

6. (a) Let (X, d) be a metric space. A set $E \subseteq X$ is called **discrete** if there is a $\delta > 0$ such that, for all x and y in E with $x =\neq y$ we have $d(x,y) > \delta$. Show that a discrete set is necessarily closed.

(b) Suppose $f:\ (0,1) \to \mathbb{R}$ is differentiable on all $(0,1)$ and $f'(1/4) < 0 < f'(3/4)$. Show that there is a $c \in (1/4, 3/4)$ such that, $f'(c) = 0$

7. Show that $\displaystyle\int_0^\infty \sin(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.