Universidad de Puerto Rico
Departamento de Matemáticas
Facultad de Ciencias Naturales
Recinto de Río Piedras

February 06, 2009

PhD Qualifying Exam: Analysis

3 hours

You may solve all eight (8) Problems (each worths 20 points) but only the best five (5) solutions will be counted as your grade. The passing grade is 70 points.

1. Let $a_1 < a_2 < \cdots < a_n < \cdots$ be the sequence of integers such that in the decimal representation of a_n, there is no digit 9. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges.

2. Assume g is a continuous real valued function on $[-\pi, \pi]$ with $g(-\pi) = g(\pi)$, and suppose that $\displaystyle\int_{-\pi}^{\pi} g(t)\sin(nt)\, dt = 0$ for all natural numbers n. Prove that g is an even function (i.e., $g(-x) = g(x)$ for all x). **Hint: Consider** $G(x) := g(x) - g(-x)$.

3. Let λ denote the Lebesgue measure on the real line.
  (a) (**10 Pts**). Prove that there is an open set $\mathcal{O}$ that is dense in $\mathbb{R}$ with $\lambda(\mathcal{O}) < 1$.
  (b) (**5 Pts**). Let $\mathcal{O}$ be any set satisfying the conclusion to part (a). Prove that $\mathbb{R} \setminus \mathcal{O}$ is **uncountable**.
  (c) (**5 Pts**). Let $\mathcal{O}$ be any set satisfying the conclusion to part (a). Prove that $\mathbb{R} \setminus \mathcal{O}$ **is not compact**.

4. (a) (**5 Pts**). State the Lebesgue Dominated Convergence Theorem for $f_n : [0, 1] \to \mathbb{R}$.
  (b) (**15 Pts**). Use (a) to evaluate $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{dx}{(1 + \frac{x}{n})^n x^{\frac{1}{n}}}$.
  [**Be sure to explain why the hypotheses are satisfied when you quote (a)**].

5. (a) (**5 Pts**). Prove that the integral $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+9}\, dx$ converges as an **improper Riemann integral**.
  (b) (**15 Pts**). Use the **calculus of residues** to evaluate the improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+9}\, dx$.

  Remark: You need to provide details to justify each step in your computation.

6. Describe a sequence of conformal maps whose composition takes
$\Big(\{z \in \mathbb{C} :\ |z-1| > 1\} \cap \{z \in \mathbb{C} :\ |z+1| > 1\}\Big) \bigcup \{\infty\}$ to $\{z \in \mathbb{C} :\ |z| < 1\}$ and ∞ to 0.

7. Let $\{f_n\}_{n=1}^{\infty}$ be analytic functions on $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$, such that there exists a constant $M > 0$ with

$$|f_n(z)| < M \quad \text{for all } \ n \in \mathbb{N} \ \text{ and } \ z \in \mathbb{D}.$$

(a) (**15 Pts**). Prove that for any $0 < \alpha < 1$, $\{f_n\}_{n=1}^{\infty}$ is an **equi-continuous family** on $\mathbb{D}_\alpha := \{z \in \mathbb{C} : \ |z| < \alpha\}$, that is, for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $z, w \in \mathbb{D}_\alpha$ and $|z - w| < \delta$ then

$$|f_n(z) - f_n(w)| < \varepsilon \ \text{ for all } \ n \in \mathbb{N}.$$

(b) (**5 Pts**). A direct consequence is that $\{f_n\}_{n=1}^{\infty}$ has a subsequence $\{f_{n_k}\}$ which converges uniformly on $\mathbb{D}_\alpha$ (**you do not need to prove this part**). Prove that the function

$$f(z) := \lim_{k\to\infty} f_{n_k}(z), \quad z \in \mathbb{D}_\alpha$$

is analytic on $\mathbb{D}_\alpha$.

8. (a) (**5 Pts**). Let f be a positive measurable function on $[0, 1]$, Give the definition of f to be Lebesgue integrable (**the definition of Lebesgue integrable of bounded measurable functions is supposed to be given and can be used**).

(b) (**15 Pts**). Let f be a positive Lebesgue integrable function on $[0, 1]$. Prove the **absolute continuity of integration**, that is, for every $\varepsilon > 0$, there is a $\delta > 0$, such that if $E \subset [0, 1]$ has Lebesgue measure $\lambda(E) < \delta$, then $\displaystyle\int_E f(x)\, dx < \varepsilon$.