Universidad de Puerto Rico
Departamento de Matemáticas
Facultad de Ciencias Naturales
Recinto de Río Piedras

September 2009

PhD Qualifying Exam: Analysis

3 hours

You may solve all seven (7) Problems (each worths 20 points) but only the best five (5) solutions will be counted as your grade. The passing grade is 70 points.

1. Let g be a nonnegative Lebesgue integrable function on $[a, b]$. Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of Lebesgue measurable functions and f is Lebesgue measurable such that
(a) $|f_n(x)| \leq g(x)$ a.e. for each $n \in \mathbb{N}$, and
(b) $\lim_{n\to\infty} f_n(x) = f(x)$ a.e.

Prove that f is Lebesgue integrable and $\displaystyle\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx$.

2. Let $E \subset [0, 1]$ has **nonzero Lebesgue measure**. Prove that there is an open interval $(\alpha, \beta) \subset [0, 1]$ such that $m(E \cap (\alpha, \beta)) > \frac{1}{2}(\beta - \alpha) > 0$.

3. Let f be a nonconstant analytic function on a connected domain $G \subset \mathbb{C}$. For any open subset $U \subset G$, prove that $f(U)$ is an open set.

4. Prove that for any $\varepsilon > 0$ and any $n \in \mathbb{N}$, there are $2(n + 1)$ positive numbers $a_1, a_2, \ldots, a_{n+1}, b_1, b_2, \ldots, b_{n+1}$ such that
(a) $a_1 + a_2 + \cdots + a_{n+1} = b_1 + b_2 + \cdots + b_{n+1}$, and

(b) $\dfrac{a_1 - b_1}{a_1 + b_1} + \dfrac{a_2 - b_2}{a_2 + b_2} + \cdots + \dfrac{a_n - b_n}{a_n + b_n} + \dfrac{a_{n+1} - b_{n+1}}{a_{n+1} + b_{n+1}} > n - \varepsilon$.

5. State Fatou's Lemma, and show that the sequence $(f_n)_{n\in\mathbb{N}}$ defined on $\mathbb{R}$ by $f_n(x) := \begin{cases} \frac{1}{n}, & \text{for } x \in [1, n] \\ 0, & \text{for } x \in \mathbb{R} \setminus [1, n] \end{cases}$ for $n \in \mathbb{N}$, provides an example where strict inequality holds.

(a) Use Fatou's Lemma to prove that $\displaystyle\int \sum_{n=1}^{\infty} g_n(x)\, dx = \sum_{n=1}^{\infty} \int g_n(x)\, dx$, where g_n is measurable and nonnegative for each $n \in \mathbb{N}$.

(b) Use the result of part (a) to show that for each $p > -1$, then $\displaystyle\int_0^1 \frac{x^p \ln(x)}{1 - x} = -\sum_{n=1}^{\infty} \frac{1}{(p + n)^2}$.

6. An entire function f is said to be periodic if there exists $z_0 \neq 0$, such that $f(z + z_0) = f(z)$ for all $z \in \mathbb{C}$. Show that if z_1 and z_2 are both periods of f, then f is constant if $\dfrac{z_1}{z_2} \notin \mathbb{R}$.

7. True or false (if true, give a short explanation, if false, give a counterexample).

(a) If an entire function is bounded on the imaginary axis, then it is bounded on all $\mathbb{C}$.

(b) If an entire function f satisfies $\lim_{z\to\infty} f(z) = c$, where c is a constant, then $f(z) = c$ for all $z \in \mathbb{C}$.

(c) If the sequence $(f_n)_{n\in\mathbb{N}}$ of entire functions converges uniformly to 0 on $|z| \leq 1$, then in fact, $f_n(z) \to 0$ for all $z \in \mathbb{C}$.

(d) Every function analytic on the open unit disk $|z| < 1$ is bounded on $|z| < 1$.