Departamento de Matemáticas
Facultad de Ciencias Naturales
Recinto de Río Piedras

February 11, 2008

PhD Qualifying Exam: Analysis

3 hours

You may solve all six (6) Problems but only the best five (5) solutions will be counted as your grade.

1. Let f_n be a sequence of nonnegative measurable functions on $[0,1]$. Suppose that f is a Lebesgue integrable function with $f_n(x) \le f(x)$ for all $x \in [0,1]$ and $n \in \mathbb{N}$. Suppose also that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

Prove that there is a subsequence f_{n_k} which converges to $f(x)$ almost everywhere.

2. Find complex numbers a, b, c, d, with $ad - bc \neq 0$, so that the Möbius transformation $f(z) := \dfrac{az+b}{cz+d}$ carries the imaginary axis to the circle whose radius is 2 and whose center is $3 = 3 + 0i$.

3. (a) Prove that for any Lebesgue measurable set $A \subset [0,1]$, we have $\displaystyle\lim_{k\to\infty} \int_A \sin(kx)\,dx = 0$.

(b) Suppose that $f(x)$ and $g(x)$ are measurable functions on $[0,1]$. Prove that $h(x) = \max\{f(x), g(x)\}$ is also measurable.

4. Use the Residue Calculus to compute $I = \displaystyle\int_0^\infty \frac{1}{(x^4+4)(x^2+9)^9}\,dx$.

To save arithmetic, you may define some **explicit** points $a_1, \ldots, a_L \in \mathbb{C}$ (what should L be?) and **explicit** functions $h_1, \ldots, h_L$, and then may express your answer explicitly in the form

$$I = \Big[h_1(a_1) + \cdots + h_L(a_L)\Big] \cdot \text{Constant}.$$

(Do not bother to perform the function-evaluation).

5. Let f and g be continuous real valued functions on $\mathbb{R}$ such that $\displaystyle\lim_{|x|\to\infty} f(x) = 0$ and $\displaystyle\int_{-\infty}^{\infty} |g(x)|\,dx < \infty$.

Define the function h on $\mathbb{R}$ by $h(x) = \displaystyle\int_{-\infty}^{\infty} f(x-y)g(y)\,dy$. Prove that $\displaystyle\lim_{|x|\to\infty} h(x) = 0$.

6. Let A and B be two sets with the properties

(a) $A \cup B = \mathbb{N} := \{1, 2, 3, \ldots\}$,

(b) $A \cap B = \emptyset$,

(c) The cardinality of A, $\text{Card}(A) = +\infty$ and the cardinality of B, $\text{Card}(B) = +\infty$.

Prove that for any real number $r > 0$, there are two sequences $a_n \in A$ and $b_n \in B$ such that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = r.$$