Universidad de Puerto Rico
Departamento de Matemáticas
Facultad de Ciencias Naturales
Recinto de Río Piedras

September 05, 2008

PhD Qualifying Exam: Analysis

3 hours

You may solve all six (6) Problems but only the best five (5) solutions will be counted as your grade.

1. (a) Construct a continuous function $f:\ \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{+\infty} |f(x)|\, dx < +\infty \qquad \text{but } \lim_{x\to\infty} f(x) \qquad \text{does not exist.}$$

(b) Suppose that $f:\ \mathbb{R} \to \mathbb{R}$ is a uniformly continuous function satisfying

$$\int_{-\infty}^{+\infty} |f(x)|\, dx < +\infty. \qquad \text{Prove that } \lim_{x\to\infty} f(x) = 0 \qquad .$$

2. Let f be an analytic function on the annulus $\{z \in \mathbb{C}:\ 0 < |z| < 1\}$ satisfying $|zf(z)| \geq 1$ for all z, and $f(1/2) = 2$. Prove that $f(z) = \dfrac{1}{z}$. **Hint: The Schwarz Lemma may be useful**.

3. Let f_n be a sequence of Lebesgue integrable functions on $\mathbb{R}$. Prove that the set $E := \{x \in \mathbb{R}:\ f_n(x) \text{ converges }\}$ is a measurable set.

4. Show that if $a > 1$, then $\displaystyle\int_0^\infty \frac{\log(x)}{x^2+a^2} = \frac{\pi}{2a}\log(a)$. **Justify all the steps**.

5. Let $I_n = \displaystyle\int_{-1}^{1}(1-x^2)^n dx$, and $p_n(x) = \dfrac{(1-x^2)^n}{I_n}$. For any continuous function $f \in C[-1,1]$, let

$$g_n(x) = \int_{-1}^{1} f(y)p_n(y-x)dy.$$

Prove that $g_n \to f$ uniformly on $[-1,1]$. (Note that g_n is a polynomial for each n.)

6. (a) Prove that the function $f(z) = -\dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$ is a conformal map from the half-disc $\mathbb{D}_+ = \{z = x+iy \in \mathbb{C}:\ |z| < 1,\ y > 0\}$ to the upper half-plane $\mathbb{H}_+ = \{z = x+iy \in \mathbb{C}:\ y > 0\}$.

(b) Let $\delta > \frac{1}{2}$ and $f_n(x) = \dfrac{x}{n^\delta(1+nx^2)}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. Prove that the series $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $\mathbb{R}$.