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Faculty of Natural Sciences
Department of Mathematics

Comprehensive Probability Test

December, 2004

Justify completely all your answers. No credit for partial answers!

To aprove the test must be unswered correctly and completely at least 5 of the 10 exercices.

1. Independent trials that result in a success with probability p and a failure with probability $1-p$ are called Bernoulli trials. Let P_n denote the probability that n Bernoulli trials result in an even number of successes (0 being considered as an even number). Show that

a) $P_n = p\,(1-P_{n-1}) + (1-p)P_{n-1}$

b) Find an expression for P_n in terms of p and n. [HINT: Use induction on n]

2. a) Prove the identity $\sum_{i=0}^{[\frac{n}{2}]} \binom{n}{2i} p^{2i} q^{n-2i} = \frac{1}{2}[(p+q)^n + (q-p)^n]$, where $q=1-p$ and $\left[\frac{n}{2}\right]$ is the largest integer less than or equal to $\frac{n}{2}$.

b) Use this identity to verify the result obtained in exercise 1. part b).

3. a) Let A and B be two events. Show that $P(A)$ is between $P(A \mid B)$ and $P(B \mid A)$.

b) Let X be any continuous random variable with cumulative distribution function $F(x)$, $x \in R$. Prove that the distribution of the random variable $F(X)$ is uniform in the interval $[0,1]$.

4. The probability that an apple tree of one particular type will produce apples in the first year after planting is 0.8. The owner of a garden center sells these to customers five trees at a time with a guarantee that if fewer than three trees produce apples in the first year he will replace all the ones that fail.

(a) Calculate the probability that no replacements will be required.

(b) If he sells five trees to each of 3000 customers, how many replacement trees can he expect to supply?

5. At work, from 8:00 a.m. to 12:00 noon, Dick and Shirley share the same bathroom. When Dick uses the bathroom, it is busy for 10 minutes, while it keeps busy just for 5 minutes when Shirley occupies it. Dick needs to occupy the bathroom at times that occurs at random with a probability that increases linearly with time, while the corresponding for Shirley decreases linearly with time. Dick needs for the bathroom are independent of those of Shirley.
Find the probability that when one of them needs the bathroom, the bathroom is already occupied. .

6. Find the distribution of the time between consecutive emissions in a Poisson(λ)experiment.

7. If X is a Poisson random variable with parameter λ, find $P\{X \ \ is \ \ even\}$ making use of the expansion of$e^{-\lambda}+e^{\lambda}$.

8. The chairman of a certain department of mathematics consisting of 100 professors, in a move to improve efficiency decides to assign a secretary for a whole day to each professor that requests it. Assume that the chances that in a certain day a professor requires the services of a secretary are 1 in 20, and that professors' need for secretaries are independent of each other. The chairman of this department, due to budget constrictions is not decided to satisfy every single professor every day. Instead, being a statistician, he thinks it will be good enough if the probability that when a professor requests a secretary there is one available is at least 0.90.
How many secretaries must be in the 'pool' to satisfy these requirements?

9. Suppose that X_1, X_2 are independent exponentially distributed random variables, and let $Y = X_1 - X_2$.

a) Find the moment generating function of Y.

b) What is the distribuction of Y.

10. Suppose that X_1, X_2 are independent exponentially distributed random variables, and let $Y = X_1 - X_2$, and $Z = X_1 + X_2$.

a) Find the joint probability distribution of Y and Z.

b) Find $E(Y \mid Z)$