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November 3, 2000 **Numerical Analysis**

SOLVE THREE OF THE FOLLOWING FIVE PROBLEMS

1.

(a) Establish the contraction mapping theorem.

(b) Derive Newton's method for the solution of $F(x) = x^2 - 5 = 0$ and carry out the first 5 iterations.

(c) Give the order of convergence of the method of b) (with proof).

2. Let A, E be real symmetric matrices. Denote by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \cdots \leq \mu_n$ the eigenvalues of A and $A+E$ respectively.

(a) Prove that $\lambda_k = \min_{V_k} \max\{x^T Ax; x \in V_k, \|x\|_2 = 1\}$ where V_k runs over all the k - dimensional subspaces of $\mathbb{R}^n$.

(b) Prove that $|\lambda_i - \mu_i| \leq \|E\|_2, \; 1 \leq i \leq n$.

(c) What conclusions do you draw about eigenvalues of real symmetric matrices?

3.

(a) Let Φ be a real-valued continuous function. Write down the divided difference interpolation formula for Φ with three points t_k, t_{k-1}, and t_{k-2} along with the remainder.

(b) Compute $\displaystyle\int_{t_k}^{t_k+h} (t-t_k)(t-t_k+h)dt$ and

$$\int_{t_k}^{t_k+h} (t-t_k)(t-t_{k-1})(t-t_{k-2})dt$$

where $t_k = t_{k-1} + h = t_{k-2} + 2h$.

(c) Consider the initial-value problem

$$(P) \begin{cases} x'(t) &= f(t, x(t)), \quad t \geq t_0 \\ x(t_0) &= x_0 \end{cases}$$

Recast the problem as an integral equation and derive the numerical scheme

$$(A) \ \ \xi_{k+1} = \xi_k + \frac{h}{12} \ \ [23 \ f_k - 16 \ f_{k-1} + 5 \ f_{k-2}]$$

to approximate the solution of (P). (Here, $f_j = f(t_j, x(t_j))$)

(d) For the method (A), define and estimate the local truncation error.

4. Consider the system $AX = b$ where A is an $n \times n$ real matrix and $b \in \mathbb{R}^n$

(a) If

$$A = \begin{bmatrix} 2 & 4 & -7 \\ 3 & 6 & -10 \\ -1 & 3 & 6 \end{bmatrix}, \qquad b = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$$

use Gaussian elimination (indicating all the steps) to solve the system.

(b) In general provide the total number of operations to carry out needed Gaussian elimination process to solve the system. (operations = additions, subtractions, multiplications, divisions)

(c) For the system in a) write down the Gauss-Jacobi method to solve it iteratively. Then taking

$$x^0 = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

carry out the first two interactions.

5. Consider the initial-value problem:

$$\left\{ \begin{array}{ll} y'(x) & = f(x, y(x)) \\ y(x_0) & = y_0 \end{array} \right\} \tag{1}$$

along with a one-step method to approximate its solution:

$$y_{n+1} = y_n + h\ F\ (x_n, y_n, h), n \geq 0 \tag{2}$$

i. Give the function F for the modified Euler method.

ii. Define (with precision) the local truncation error for the method (2).

Assume henceforth that the following conditions are satisfied:

$$F(x, y, 0) = f(x, y) \tag{3}$$

$$F_h(x, y, 0) = \frac{1}{2}\left[f_x(x, y) + f_y(x, y)f(x, y)\right] \quad (4)$$

where the subscript denotes the partial derivative with respect to the corresponding variable.

iii. Show that the local truncation error has order 3.

iv Show that the above conditions are satisfied by the modified Euler method.

v. Give the general expression of an explicit Runge-Kutta method with two slopes.

vi. Give the conditions on the parameters of the method in v) in order for the result in iii) to apply.