May 1, 2003

Linear Programming

SOLVE EXACTLY THREE OUT OF THE FOLLOWING FIVE PROBLEMS:

1. Given a linear programming problem in standard form

\[
\begin{align*}
\text{minimize} & \quad c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\
\text{subject to} & \quad A(x_1, x_2, \ldots, x_n)^T = (b_1, b_2, \ldots, b_m)^T \\
& \quad x_i \geq 0 \quad 1 \leq i \leq n
\end{align*}
\]

Prove that: If there is an optimal feasible solution, there is an optimal basic feasible solution. (\(A\) is a real \(m \times n\) matrix)

2. Solve the following problem.

\[
\begin{align*}
\text{minimize} & \quad -2x_1 + 4x_2 + 7x_3 + x_4 + 5x_5 \\
\text{subject to} & \quad -x_1 + x_2 + 2x_3 + x_4 + 2x_5 = 7 \\
& \quad -x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 6 \\
& \quad -x_1 + x_2 + x_3 + 2x_4 + x_5 = 4 \\
& \quad x_1 \text{ free, } x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0
\end{align*}
\]
3. In the standard linear programming problem 1, suppose \( n > m \) and \( A \) has rank \( m \). Suppose the first \( m \) columns of \( A \) is linearly independent. Let \( B \) be the \( m \times m \) matrix consisted of the first \( m \) columns of \( A \). So \( A = [B, D] \), \( \vec{x} = (x_B, x_D) \), \( c = (c_B, c_D) \)

2 points a) What is the relative cost vector?

3 points b) If we choose \( \vec{x}_0 = (B^{-1} \vec{b}, \vec{0}_D) \) solution, how to determine if it is optimal?

5 points c) If \( \vec{x}_0 \) is not optimal, how to determine which column of \( A \) to enter and which column of \( A \) to leave the basis?

4. For the problem

\[
\text{minimize} \quad 2x_1 + x_2 + 4x_3 \\
\text{subject to} \quad x_1 + x_2 + 2x_3 = 3 \\
\quad \quad \quad \quad \quad 2x_1 + x_2 + 3x_3 = 5 \\
\quad \quad \quad \quad \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
\]

3 points a) What is the dual problem.

7 points b) Note that \( \lambda = [1, 0] \) is feasible for the dual, starting with this \( \lambda \). Solve the primal using the primal-dual algorithm.

5. Convert the following problem to a linear program in standard form

\[
\text{minimize} \quad |x| + |y| + |z| \\
\text{subject to} \quad x + y \leq 1 \\
\quad \quad \quad \quad \quad 2x + z = 3
\]