University of Puerto Rico
Río Piedras Campus
Department of Mathematics

Functional Analysis I
PhD Qualifying Exam

Monday, February 11, 2008

Each problem is worth 25 points. Only the best four solutions will be counted. The passing score is 60 points or higher. Time: 3 hours

Note. For a Hilbert space H with inner product $< .,. >$, we denote by $\mathcal{B}(H)$ the algebra of all bounded linear operators on H. Recall that $H \oplus H$ is the Hilbert space consisting of all pairs (u, v).

(1) (a) Let X be a Banach space. Show that if X^* (the dual of X) is separable, then X is separable.

(b) Prove that $l^\infty(\mathbb{N})$ is a nonseparable Banach space.

(2) (a) Let X be a Banach space and $T \in \mathcal{L}(X)$ be a compact operator. Prove that if $\lambda \in \sigma(T) \setminus \{0\}$, then λ is an eigenvalue of T.

(b) Suppose that (T_n) is a sequence of bounded linear operators on a Banach space X such that there exists a bounded linear operator T such that $\lim_{n\to\infty} T_n x = Tx$ for all $x \in X$. Show that for any compact operator K on X, $\lim_{n\to\infty} \|T_n K - TK\| = 0$.

(3) (a) Let X be a normed space and $T : X \longrightarrow X$ be a linear map. Prove that if for any sequence (x_n) converging to 0, the sequence (Tx_n) is bounded, then T is continuous.

(b) Suppose T is a bounded linear operator on a Banach space X such that there exists $\lambda \in \mathbb{C}$ and a sequence x_n with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}(Tx_n - \lambda x_n) = 0$. Prove that $\lambda \in \sigma(T)$.

(4) (a) Let X, Y be Banach spaces and T a linear map from X to Y. Prove that T is continuous from X endowed with the weak topology $\sigma(X, X^*)$ to Y endowed with the norm topology, then T has finite rank.

(b) Find a Hilbert space H and a subspace F of X such that $H \neq F \oplus F^T$.

(5) (a) Let $X = l^2(\mathbb{N})$ and $C = \{0\} \cup \{\frac{1}{n}e_n, n \in \mathbb{N}\}$.

(i) Prove that C is compact.

(ii) Is the convex hull of C compact?

(b) Prove that every weakly compact subset of a Banach space is norm bounded.

(6) (a) Let H be a Hilbert space and (x_n) a sequence in H. Suppose that $x \in H$ and $\lim_{n\to\infty} < x_n, v >=< x, v >$ for all $v \in H$. Assume that $\lim_{n\to\infty} \|x_n\| = \|x\|$. Prove that $\lim\limits_{n\to\infty} x_n = x$ in H.

(b)