University of Puerto Rico
Río Piedras Campus
Department of Mathematics

Functional Analysis I
Ph.D. Qualifying Exam

Thursday, September 6, 2007

There are five problems. Each problem is worth 25 points. Only the best four solutions will be counted. The passing score is 60 points or higher. Time: 3 hours

Note. For a Hilbert space H with inner product $< .,. >$, we denote by $\mathcal{B}(H)$ the algebra of all bounded linear operators on H. Recall that $H \oplus H$ is the Hilbert space consisting of all pairs $(u, v) \in H \times H$ with the inner product $< (u_1, v_1), (u_2, v_2) >=< u_1, u_2 > + < v_1, v_2 >$.

(1) (a) Let H be a separable infinite dimensional Hilbert space. Show that for any bounded linear operator T on H, there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of bounded linear finite rank operators on H such that $\lim_{n\to\infty} T_n x = Tx$ for every $x \in H$.

(b) Let $H = l^2(\mathbb{N})$ and consider the operator T on H given by $T(x_1, x_2, ..., x_n, ...) = (ix_1, i^2 x_2, ..., i^n x_n, ...)$, where $i^2 = -1$. Does there exist a sequence $(T_n)_{n\in\mathbb{N}}$ of bounded linear finite rank operators on H such that $\lim_{n\to\infty} \|T_n - T\| = 0$?

(2) (a) Let X be a Banach space and $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in X^* such that $\sum_{n=1}^{\infty} \varphi_n(x)$ converges for every $x \in X$. Prove that the series $\sum_{n=1}^{\infty} \frac{\|\varphi_n\|}{2^n}$ is convergent.

(b) Let H be an infinite dimensional Hilbert space. Show that there exists an operator $T \in \mathcal{B}(H) \setminus \{0\}$ such that $T^2 = 0$.

(3) (a) Prove that a normed linear space X is complete if and only if every absolutely convergent series in X is convergent. (A series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if $\sum_{n=1}^{\infty} \|x_n\|$ is convergent).

(b) Let H be a Hilbert space and let $u \in \mathcal{B}(H)$ be such that $u = u^*$ and $u^2 = 1$. Show that there exists a projection p in $\mathcal{B}(H)$ such that $u = 2p - 1$. (A projection p in $\mathcal{B}(H)$ is an element of $\mathcal{B}(H)$ with $p^2 = p = p^*$).

(4) (a) Let X be a normed space and Y a closed subspace of X. Show that if Y and X/Y are separable, then X is separable.

(b) Let H be a Hilbert space, $A \in \mathcal{B}(H)$ and define B on $H \oplus H$ by $B = \begin{pmatrix} 0 & iA \\ -iA^* & 0 \end{pmatrix}$. Prove that B is self-adjoint and $\|A\| = \|B\|$. (Here, $\|B\|$ is the norm of B as an element of $\mathcal{B}(H \oplus H)$).

(5) Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be such that $\|T\| \leq 1$. Let $x \in H$ be such that $Tx = x$. Prove that $T^*x = x$.