There are five problems. Each problem is worth 25 points. Only the best four solutions will be counted. The passing score is 60 points or higher. Time: 3 hours

Note. For a Hilbert space $H$ with inner product $<\cdot,\cdot>$, we denote by $\mathcal{B}(H)$ the algebra of all bounded linear operators on $H$. Recall that $H \oplus H$ is the Hilbert space consisting of all pairs $(u, v) \in H \times H$ with the inner product $<(u_1, v_1), (u_2, v_2)> = <u_1, u_2> + <v_1, v_2>.$

1. (a) Let $H$ be a separable infinite dimensional Hilbert space. Show that for any bounded linear operator $T$ on $H$, there exists a sequence $(T_n)_{n \in \mathbb{N}}$ of bounded linear finite rank operators on $H$ such that $\lim_{n \to \infty} T_n x = Tx$ for every $x \in H$.

   (b) Let $H = l^2(\mathbb{N})$ and consider the operator $T$ on $H$ given by $T(x_1, x_2, ..., x_n, ...) = (ix_1, i^2x_2, ..., i^n x_n, ...)$, where $i^2 = -1$. Does there exist a sequence $(T_n)_{n \in \mathbb{N}}$ of bounded linear finite rank operators on $H$ such that $\lim_{n \to \infty} \|T_n - T\| = 0$?

2. (a) Let $X$ be a Banach space and $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence in $X^*$ such that $\sum_{n=1}^{\infty} \varphi_n(x)$ converges for every $x \in X$. Prove that the series $\sum_{n=1}^{\infty} \frac{\|\varphi_n\|}{2^n}$ is convergent.

   (b) Let $H$ be an infinite dimensional Hilbert space. Show that there exists an operator $T \in \mathcal{B}(H) \setminus \{0\}$ such that $T^2 = 0$.

3. (a) Prove that a normed linear space $X$ is complete if and only if every absolutely convergent series in $X$ is convergent. (A series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if $\sum_{n=1}^{\infty} \|x_n\|$ is convergent).

   (b) Let $H$ be a Hilbert space and let $u \in \mathcal{B}(H)$ be such that $u = u^*$ and $u^2 = 1$. Show that there exists a projection $p$ in $\mathcal{B}(H)$ such that $u = 2p - 1$. (A projection $p$ in $\mathcal{B}(H)$ is an element of $\mathcal{B}(H)$ with $p^2 = p = p^*$).
(4) (a) Let $X$ be a normed space and $Y$ a closed subspace of $X$. Show that if $Y$ and $X/Y$ are separable, then $X$ is separable.

(b) Let $H$ be a Hilbert space, $A \in \mathcal{B}(H)$ and define $B$ on $H \oplus H$ by $B = \begin{pmatrix} 0 & iA \\ -iA^* & 0 \end{pmatrix}$. Prove that $B$ is self-adjoint and $\|A\| = \|B\|$. (Here, $\|B\|$ is the norm of $B$ as an element of $\mathcal{B}(H \oplus H)$).

(5) Let $H$ be a Hilbert space and $T \in \mathcal{B}(H)$ be such that $\|T\| \leq 1$. Let $x \in H$ be such that $Tx = x$. Prove that $T^*x = x$. 
