MS Qualifying Examination: Complex Analysis

Choose any three (3) of the following five (5) Problems.

Time for the Examination: Three (3) Hours

1. For each of the following functions, find (if possible) the power series expansion about \( z_0 \) and give (if possible) the radius of convergence.

   (a) (3 points) \( f(z) = \frac{1}{z} \) with \( z_0 = 2 \).
   
   (b) (3 points) \( g(z) = z + \pi \) with \( z_0 = 1 \).
   
   (c) (4 points) \( h(z) = \cosh^2(z) \) with \( z_0 = 0 \).

2. Classify all isolated singularities of the following functions. Calculate the residue at each singularity.

   (a) (3 points) \( f(z) = \frac{z^2}{(1 + z^2)^2} \).
   
   (b) (3 points) \( g(z) = \frac{\sinh(z)}{\sin(z)} \).
   
   (c) (4 points) \( h(z) = \frac{\sin(z)}{z^7} \).

3. (a) (4 points) Suppose that \( f \) and \( g \) are entire functions, and that \( |f(z)| \leq |g(z)| \) for all \( z \in \mathbb{C} \). Show that if \( z = 0 \) is the only zero of \( g \), then there is a constant \( C \) such that \( f(z) = Cg(z) \) for all \( z \in \mathbb{C} \).

   (b) (6 points) Let \( f \) be an analytic function on \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) and suppose that \( |f(z)| \leq 1 \) for all \( z \in \mathbb{D} \). Show that \( |f'(0)| \leq 1 \).

4. (a) (2 points) State a version of the residue theorem.

   (b) (2 points) Find the residues of the function \( f(z) = \frac{e^{iz}}{(z - 1)^2 + 4} \) at each isolated singularity.

   (c) (6 points) Evaluate the indefinite integral
   
   \[ \int_{-\infty}^{\infty} \frac{\cos(x)}{(x - 1)^2 + 4} \, dx. \]

5. (a) (4 points) Let \( f \) be an entire function with \( \text{Re}(f(z)) \geq 1 \) for all \( z \in \mathbb{C} \). Prove that \( f \) is constant.

   (b) (3 points) Show that if \( f \) and \( g \) are analytic functions on a region \( G \) such that \( fg \) is analytic then either \( f \) is constant or \( g \equiv 0 \).

   (c) (3 points) Find the maximal domain of analyticity of the function \( f(z) = \frac{1}{2i} \log \left( \frac{1 + iz}{1 - iz} \right) \).