February 6, 2007

MS Qualifying Examination: Complex Analysis

Choose any three (3) of the following five (5) Problems.

Time for the Examination: Three (3) Hours

1. For each of the following functions, find (if possible) the power series expansion about z_0 and give (if possible) the radius of convergence.

   (a) (3 points) $f(z) = \dfrac{1}{z}$ with $z_0 = 2$.

   (b) (3 points) $g(z) = z + \overline{z}$ with $z_0 = 1$.

   (c) (4 points) $h(z) = \cosh^2(z)$ with $z_0 = 0$.

2. Classify all isolated singularities of the following functions. Calculate the residue at each singularity.

   (a) (3 points) $f(z) = \dfrac{z^2}{(1+z^2)^2}$.

   (b) (3 points) $g(z) = \dfrac{\sinh(z)}{\sin(z)}$.

   (c) (4 points) $h(z) = \dfrac{\sin(z)}{z^7}$.

3. (a) (4 points) Suppose that f and g are entire functions, and that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that if $z = 0$ is the only zero of g, then there is a constant C such that $f(z) = Cg(z)$ for all $z \in \mathbb{C}$.

   (b) (6 points) Let f be an analytic function on $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ and suppose that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Show that $|f'(0)| \leq 1$.

4. (a) (2 points) State a version of the residue theorem.

   (b) (2 points) Find the residues of the function $f(z) = \dfrac{e^{iz}}{(z-1)^2+4}$ at each isolated singularity.

   (c) (6 points) Evaluate the indefinite integral

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{(x-1)^2+4}\, dx.$$

5. (a) (4 points) Let f be an entire function with $\mathrm{Re}(f(z)) \geq 1$ for all $z \in \mathbb{C}$. Prove that f is constant.

   (b) (3 points) Show that if f and g are analytic functions on a region G such that $\overline{f}g$ is analytic then either f is constant or $g \equiv 0$.

   (c) (3 points) Find the maximal domain of analyticity of the function $f(z) = \dfrac{1}{2i} \log\left(\dfrac{1+iz}{1-iz}\right)$.