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MS Qualifying Exam- Complex Variables  

Solve any three of the following five problems

(1) (a) Let $U$ be an open subset of $\mathbb{C}$, and $a \in U$ be an isolated singularity for a holomorphic function $f : U \rightarrow \mathbb{C}$. Prove that if $a$ is not a pole of $f$, then $f(U \setminus \{a\}) = \mathbb{C}$.

(b) Evaluate the integral $\int_{\gamma} z^2 + \frac{1}{z(z^2 + 4)} \, dz$ where $\gamma(t) = re^{it}$, $0 \leq t \leq 2\pi$, for all values of $r > 0$, $r \neq 2$.

(2) (a) State an prove the theorem of Goursat.

(b) Prove that $\int_{0}^{\infty} \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{4a^3}$ where $a > 0$, by the method of residues.

(3) (a) Suppose $\gamma_0$ and $\gamma_1$ are closed rectifiable curves (with parameter interval $[0, 1]$) in the complex plane and there exists $\alpha \in \mathbb{C}$ such that for all $t \in [0, 1]$, $|\gamma_0(t) - \gamma_1(t)| < |\alpha - \gamma_0(t)|$. Prove that $n(\gamma_0, \alpha) = n(\gamma_1, \alpha)$ where $n(\gamma, a)$ is the index of the curve $\gamma$ with respect to the point $a$.

Hint. Consider the path $\gamma := (\gamma_1 - \alpha)/(\gamma_0 - \alpha)$ and first verify in which component of $\mathbb{C} \setminus \gamma$ the point 0 is located.

(b) Let $a \in \mathbb{R} \setminus \{0\}$. Compute the integral $\int_{0}^{\infty} \frac{x^2 \, dx}{(x^2 + a^2)^3}$ by the method of residues.

(4) (a) Show that the sum of a power series is holomorphic in the interior of its circle of convergence.

(b) Compute $\int_{0}^{2\pi} \frac{d\theta}{1 - 2\alpha \cos \theta + \alpha^2}$ where $|\alpha| \neq 1$.

(5) (a) Let $f$ be an entire function such that $\lim_{|z| \to \infty} |f(z)| = \infty$. Prove that $f$ is a polynomial.

(b) Compute the integral $\int_{0}^{\pi/2} \frac{dt}{a^2 \cos^2 t + b^2 \sin^2 t}$ by considering the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a, b > 0$.  

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