Universidad de Puerto Rico
Departamento de Matemáticas

December 1, 2004

M.S. Qualifying Examination

Complex Variables:

Solve any **three** of the following five problems
(each problem has two parts).

(1) (a) Suppose $a \in \Omega$ where $\Omega \subset \mathbb{C}$ is open. Let $f : \Omega \setminus \{a\} \longrightarrow \mathbb{C}$ be analytic. Show that f has an essential singularity at $z = a$ if and only if for all $s \in \mathbb{R}$, $\lim_{z \to a} |z-a|^s |f(z)|$ does not exist in $[0, \infty]$.

(b) Find the Laurent series expansion of $f(z) = \dfrac{z+1}{z(z-1)(z-2)}$ in the annulus $1 < |z| < 2$.

(2) (a) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be continuous. Suppose that f is analytic on $\mathbb{C} \setminus [0, 1]$. Prove that f is an entire function.

(b) Compute $\displaystyle\int_0^\infty \frac{x}{(1+x^2)^2} dx$.

(3) (a) Let $\Omega \subset \mathbb{C}$ be open and simply connected and $f : \Omega \longrightarrow \mathbb{C}$ be analytic and with $f(a) = 1$ for some $a \in \Omega$. Prove that $A := \{z \in \Omega,\ f(z) = 0\}$ is countable.

(b) Compute $\displaystyle\int_{|z|=10} \frac{z+1}{e^z+1} dz$ where integration is performed clockwise.

(4) (a) Let (P_n) be a sequence of polynomials converging uniformly on $\{z : |z| = 1\}$. Prove that (P_n) converges uniformly on $\{z : |z| \leq 1\}$.

(b) Compute the integral $\displaystyle\int_0^\infty \frac{x^2 \cos x}{1+x^4} dx$.

(5) (a) Let the open set $\Omega \subset \mathbb{C}$ be simply connected and $f : \Omega \longrightarrow \mathbb{C}$ be analytic. Suppose $|f(z)|$ is constant. Prove that f must be constant.

(b) Prove that all five roots of the polynomial $g(z) = z^5 + 3z + 1$ lie in the disk $\{z : |z| < 2\}$.