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PH.D ALGEBRA QUALIFYING EXAM

1. This test is divided in two (2) parts: *Group Theory* and *Ring Theory*. Each part consists of four (4) problems. The test has a total of eight (8) problems.
2. Turn off the cell phone and any other electronic decive.
3. Show your work. To get credit, your answers must be well-written, well-organized, and properly justified.
4. To pass, you should get at least fifteen (15) points in each part AND forty (40) points overall.
5. This is a three (3) hour test.

PART I: Group Theory

1. Do the following:

   (a) Suppose that G is a group. Let G act on a set X. Denote this action by $g \cdot x$ where $g \in G$ and $x \in X$. Let $\mathrm{Stab}(x)$ be the stabilizer of x under the group action. Show that stabilizers of elements in the orbit of x are conjugate subgroups. Explicitly, for every $g \in G$ and $x \in X$ we have **(5 pts)**

   $$\mathrm{Stab}(g \cdot x) = g\,\mathrm{Stab}(x)\,g^{-1}.$$

   (b) Let G act on the set X. Show that for a given $x \in X$, $\Phi(g\mathrm{Stab}(x)) = g \cdot x$ is a well-dened injective mapping of the set of left cosets of $\mathrm{Stab}(x)$ into X, and is bijective if the action is transitive. **(5 pts)**

2. A subgroup H of a group G is called *characteristic* if the image of H under any automorphism of G is itself H. Suppose that G is a group such that $|G| = 1862 = 2 \cdot 7^2 \cdot 19$. Let P be a Sylow 7-subgroup. Show that P is a characteristic subgroup of G. **(10 pts)**

3. Let G be a group of order $3393 = 9 \cdot 13 \cdot 29$. Use Sylow's Theorems to prove that G is not simple. **(10 pts)**

4. Prove that every finite group is isomorphic to a subgroup of A_n (the alternating group on n letters) for some $n \in \mathbb{N}$. **(10 pts)**

PART II: Ring Theory

1. Let $f : R \to S$ be a ring homomorphism, with R and S commutative.

   (a) If P is a prime ideal of S, show that the preimage $f^{-1}(P)$ is a prime ideal of R. **(6 pts)**

   (b) If M is a maximal ideal of S, prove or disprove that the preimage $f^{-1}(M)$ is a maximal ideal of R. **(4 pts)**

2. Do the following:

   (a) A field C is called *algebraically closed* if <u>every</u> polynomial $f(x)$ with coefficients in C has a root α in C, i.e. $f(\alpha) = 0$. Show that every algebraically closed field is infinite. **(4 pts)**

   (b) Prove or disprove the following: $x^4 + x^3 + x + 2$ is irreducible in the finite fields of three elements. **(6 pts)**

3. Let $R = \mathbb{C}[x, y]$ be the ring of polynomials in the variables x and y, so R may be viewed as $\mathbb{C}$-valued functions on complex 2-space, $\mathbb{C}^2$, in the usual way ($\mathbb{R}$ is called the coordinate ring of this space). Let I be the ideal of all functions in R that vanish on both coordinate axes, i.e., that are zero on the set $\{(a, 0) \mid a \in \mathbb{C}\} \cup \{(0, b) \mid b \in \mathbb{C}\}$. (You may assume I is an ideal.)

   (a) Exhibit a set of generators for I. (Be sure to explain why they generate I.) **(6 pts)**

   (b) Show that I is not a prime ideal. **(2 pts)**

   (c) Show that R/I has no nilpotent elements. **(2 pts)**

4. For a ring R, let $\mathrm{Spec}(R)$ be the set of all prime ideals of R. Let $\mathbb{F}$ be a field. Let $f(X) = X^2 + 7X + 7 \in \mathbb{F}[X]$. Provide an example of a field $\mathbb{F}$ with smallest possible cardinality $|\mathbb{F}|$ such that

   (a) $f(X)$ has no roots in $\mathbb{F}$. **(4 pts)**

   (b) $f(X)$ has a root of multiplicity two in $\mathbb{F}$ **(3 pts)**

   (c) $f(X)$ has two distinct roots in $\mathbb{F}$ **(3 pts)**

   Justify your assertions. Compute $\mathrm{Spec}(\mathbb{F}[X]/(f(X)))$ in each of the three cases.