Algebra Qualifying Exam (Feb 15, 2008)

Graduate Program

Department of Mathematics

University of Puerto Rico-Río Piedras (UPR-RP)

Instructions for master' students: You can choose three problems from the first five. Each problem carries 10 points. Only three problems are going to be graded. Justify all your answers.

Instructions for Ph. D' students: There are nine problems. Each problem carries 10 points. You can choose six problems from the nine problems. Six problems will be graded. You must justify all your answers. A necessary condition to pass the exam is to solve a problem correctly in each area; (this does not guarantee passing the exam).

We use the following notation:

1. $\mathbb{Q}$-rational numbers
2. $\mathbb{R}$-real numbers

1. (Linear algebra) Let

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 4 & 3 \\ -1 & -2 & -1 \end{pmatrix}$$

   a) Compute the characteristic polynomial $c_A(x)$ and the minimal polynomial $m_A(x)$ of A.

   b) Determine the eigenvalues of the matrix A and their corresponding eigenspaces.

   c) Determine the Jordan canonical form of the matrix A.

2. (Group theory) Let G be a group with normal subgroup N and subgroups $K \trianglelefteq H \trianglelefteq G$.

   a) Prove that HN/KN is isomorphic with a quotient grupo of of H/K

   b) Prove that $(H \cap N)/((K \cap N)$ is isomorphic with a subgroup of H/K

   c) If H/K is nontrivial, prove that at least one of

$$HN/KN \text{ or } (H \cap N)/((K \cap N)$$

   must be nontrivial.

3. (Group theory) Let G be a group of order 24 having no normal subgroups of order 3. Show that G has four subgroups of order six.

4. (Ring theory) Let $R \neq \{0\}$ be an integral domain such that the polynomial ring $R[x]$ is a principal ideal domain. Prove that R must be a field.

5. (Ring theory) Let α, β be complex numbers satisfying the relations: $\alpha^3 + 6\alpha^2 + 1 = 0$ and $\beta = \alpha^7 + 6\alpha^6 + \alpha^4 + \alpha^3 + 7\alpha^2 + 2$. Show that $\beta \neq 0$ and determine $1/\beta$ in $\mathbb{Q}[\alpha]$

6. (Linear algebra) Let A be a real $n \times n$-matrix and let b be a real $1 \times n$-matrix. The Rouché-Frobenius theorem says that the system $AX = b$ is compatible if and only if A and the extended matrix $[A, b]$ have the same rank; if that common rank is n, then the system has a unique solution. Using the Rouché-Frobenius theorem, what can you say about the compatibility of the following system of equations according to the value of the real parameter a?

$$a^2x + ay + z = 1$$
$$x + ay + z = a$$
$$x + ay + a^2z = 1$$

7. (Linear algebra) Let S and T be subspaces in $\mathbb{R}^4$ given by

$$S = \{(x, y, z, t)|x+y+z+t = 0, 2x-y+2z-t = 0, 4x+y+4z+t = 0\}$$

$$T = \{(x, y, z, t)|x = a + b + 2c, y = b + c, z = -a + b, t = 3b + 3c\}$$

Find a basis and the dimension of $S, T, S + T$ and $S \cap T$.

8. (Group theory) State Sylow's theorems for finite groups. Prove that a group of order 105 cannot be simple. In fact, show that a group G of order 105 has a cyclic normal subgroup of order 35, and that G is solvable. If G contains a normal subgroup of order 3, show that G is abelian.

9. (Ring Theory) Construct a field of order 16 as an extension of a field of order 4 and determine the order of each one of its nonzero elements.