Algebra Qualify Exam
(Graduate Program)

Department of Mathematics

University of Puerto Rico (UPR)

There are nine problems. Each problem carries 10 points. You can choose six from the nine problems. Only six problems are going to be graded. Justify all your answers.

For the Ph.D. students:*i) A necessary condition to pass the exam is to solve a problem correctly in each area (this does not guarantee to pass the exam); ii) The minimum score to past the exam is at least 40 points.*

For the M.S. Students: *The level of passing score will be a score of 20 from the three highest scored problems, each from the three different areas.*

We use the following notation:

1. $\mathbb{Z}$-integers
2. $\mathbb{Q}$-rational numbers
3. $\mathbb{R}$-real numbers
4. $\mathbb{Z}_n$-integers modulo n.

1.Linear Algebra : (6+4 points) Let t_0, t_1, t_2 be three distinct real numbers. Find the determinant $det(A)$, of the matrix $A = (a_{ij})$ where $a_{ij} = t_i^j$, $0 \leq i, j \leq 2$. Show that $det(A) \neq 0$.

2.Ring Theory: Determine irreducible polynomials of degree ≤ 4 in the ring $\mathbb{F}_2[X]$, where F_2 is the field $\{0, 1\}$.

3.Group Theory: Suppose G is finite group and $G = H \cup K \cup L$ for proper subgroups H, K and L. Show that $|G : H| = |G : K| = |G : L| = 2$.

4.Group Theory: Let $|G| = 2m$, where m is odd. Prove that G has a normal subgroup of order m.

5.Ring Theory: Determine the fixed field of the automorphism $t \to t + 1$ of $\mathbb{K}(t)$, where $\mathbb{K}$ is field of characteristic p and t is an indeterminate.

6.Group Theory: Let F be a field with p^m elements where p is prime.

(a) (4 points) Let G be the group G of of invertible 2×2 matrices with entries in F, and H its subgroup of matrices of determinant 1. Determine the order of H.

(b) (3+3 points) Find the order of its p-Sylow subgroups G. Show that any p-Sylow subgroup is isomorphic to the additive group of F.

7.Ring Theory: Let $\mathbb{E}^+$ be the set of even positive integers. A number is prime within $\mathbb{E}^+$ if it cannot be factored within $\mathbb{E}^+$. For instance, 6 is prime within $\mathbb{E}^+$; sure, $6 = 2 \cdot 3$, but 3 is not in $\mathbb{E}^+$. On the other hand, 4 is not prime within $\mathbb{E}^+$ because $4 = 2 \cdot 2$.

(a) (4 pts) Prove that every number in $\mathbb{E}^+$ has a prime factorization within $\mathbb{E}^+$.

(b) (2 pts)Find a number in $\mathbb{E}^+$ that has 2 different prime factorizations within $\mathbb{E}^+$.

(c) (4 pts) Determine with proof the set of all numbers in $\mathbb{E}^+$ that have two different prime factorizations within $\mathbb{E}^+$.

8.Linear Algebra: Consider the matrix A

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}$$

(a) (2+3 pts) Find the characteristic and minimal polynomials of A.

(b) Find the Jordan Canonical form of A.

9.Linear Algebra: Suppose P is the change of basis matrix from a basis $\{u_i\}$ to a basis $\{w_i\}$, and suppose Q is the change of basis matrix from basis $\{w_i\}$ back to the basis $\{u_i\}$. Prove that P is invertible and $Q = P^{-1}$.